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The *q*-Riccati Algebra

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Abstract: For $q\in(0, 1)$, we introduce the q-Riccati Lie algebra. Using the q-derivative (or Jackson derivative), we give a representation of this Lie algebra.

Keywords: q-Derivative, q-Riccati Lie Algebra

Introduction

In the mathematical field of representation theory, the representation of a Lie algebra is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator. More precisely, a representation of a Lie algebra g is a linear transformation:

$$\psi : g \to M(V)$$

where, $M(V)$ is the set of all linear transformations of a vector space V. In particular, if $V = \mathbb{R}^n$, then $M(V)$ is the set of $n \times n$ square matrices. The map ψ is required to be a map of Lie algebras so that:

$$\psi\left[(A,B)\right]=\psi(A)\psi(B)-\psi(B)\psi(A)$$

for all $A, B \in g$. Note that the expression AB only makes sense as a matrix product in a representation. For example, if A and B are antisymmetric matrices, then AB-BA is skew-symmetric, but AB may not be antisymmetric. The possible irreducible representations of complex Lie algebras are determined by the classification of the semi simple Lie algebras. Any irreducible representation V of a complex Lie algebra g is the tensor product $V = V_0 \otimes L$, where V_0 is an irreducible representation of the quotient $g_{ss}|Rad(g)$ of the algebra g and its Lie algebra radical and L is a one-dimensional representation. In the study of representations of a Lie algebra, a particular ring, called the universal enveloping algebra, associated with the Lie algebra plays an important role. The Riccati algebra is a finite-dimensional linear space that is closed under commutator, that is R is a Lie algebra.

In recent years the q-deformation of the Heisemburg commutation relation has drawn attention. Leeuwen and Maassen (1995) and many of other researcher like (Altoum, 2018a; 2018b; Rguigui, 2015a; 2015b; 2016a; 2016b; 2018a; 2018b; Altoum *et al*., 2017), the purpose is to study the probability distribution of a non-commutative random variable $a + a^*$, where a is a bounded operator on some Hilbert space satisfying:

$$aa^* - qa^*a = 1, \tag{1}$$

for some $q \in [-1, 1)$. The calculation is inspired by the case, $q = 0$, where a and a^* turn out to be the left and right shift on $l^2(\mathbf{N})$: In this case a and a^* can be quite nicely represented as operators on the Hardy class $\mathcal{H}^2$ of all analytic functions on the unit disk with L^2 limits toward the boundary. Subsequently, they find a measure μ_q, $q\in[0, 1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: μ_q is concentrated on a family of concentric circle, the largest of which has the radius $\frac{1}{\sqrt{1-q}}$. Their representation space (Leeuwen and Maassen, 1995) will be $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$, the completion of the analytic functions on

$\mathfrak{D}_q = \left\{ z \in \mathbb{C} \mid |z|^2 < \frac{1}{(1-q)} \right\}$ with respect to the inner product defined by μ_q. In this space annihilation operator a is represented by a q difference operator D_q. As q tends to 1, μ_q will tend to the Gauss measure on $\mathbb{C}$ and D_q becomes differentiation. We recall some basic notations of the language of q-calculus (Abdi, 1962; Adams, 1929; Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995). For $q\in(0, 1)$ and analytic f: $\mathbb{C} \to \mathbb{C}$ define operators Z and D_q as follows (Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995):

$$(Zf)(z) := zf(z),$$
$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, z \neq 0 \\ f'(0) \end{cases}$$

In this paper, we introduce the q-Riccati Algebra. This paper is organized as follows: In Section 1, we present preliminaries include q-calculus. In Section 2, we introduce the q-Riccati algebra. In section 3, we give a representation of this algebra.

Representation of the q-Riccati Algebra

Let $q\in(0, 1)$. Then, we define the q-Riccati Lie algebra as follows:

$$R_q = \langle A, B, C, D \rangle$$

such that:

1. $[A, B] = AD$.
2. $[A,C] = [2]_q CD$.
3. $[B,C] = qCD$.
4. $[A, D] = 0$.
5. $[B, D] = (1-q)BD$.
6. $[C, D] = (1-q)[2]_q CD$.

Representation of the q-Riccati Algebra

Let $M_{0,q}$, $M_{1,q}$ and $M_{2,q}$ given by:

$$M_{0,q} = D_q$$
$$M_{1,q} = XD_q$$
$$M_{2,q} = X^2 D_q$$

where, D_q and X are defined as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}$$
$$Xf(x) = xf(x).$$

Proposition 3.1

For $q\in(0, 1)$ we have:

i) $[M_{0,q}, M_{1,q}] = M_{0,q}H_q$
ii) $[M_{0,q}, M_{2,q}] = [2]_q M_{1,q}H_q$
iii) $[M_{1,q}, M_{2,q}] = qM_{2,q}H_q$

where, H_q is given by $H_q f(x) = f(qx)$

Proof

We have:

$$\left[M_{0,q}, M_{1,q}\right] = \left[D_q, XD_q\right]$$
$$= D_q XD_q - XD_q D_q$$

But:

$$D_q XD_q f(x) = D_q\left(x\frac{f(x) - f(qx)}{x(1-q)}\right)$$
$$= \frac{1}{1-q} D_q\left(f(x) - f(qx)\right)$$
$$= \frac{1}{1-q}\frac{f(x) - f(qx) - f(qx) + f(q^2x)}{x(1-q)}$$
$$= \frac{1}{1-q}\frac{f(x) - 2f(qx) + f(q^2x)}{x(1-q)}$$

and:

$$XD_q D_q f(x) = xD_q\left(\frac{f(x) - f(qx)}{x(1-q)}\right)$$
$$= \frac{x}{1-q}\left(\frac{\frac{f(x) - f(qx)}{x} - \frac{f(qx) + f(q^2x)}{qx}}{x(1-q)}\right)$$
$$= \frac{1}{1-q}\frac{qf(x) - qf(qx) - f(qx) + f(q^2x)}{qx(1-q)}.$$

Then, we obtain:

$$\left[M_{0,q}, M_{1,q}\right] f(x) = \frac{f(qx)(1-q) - (1-q)f(q^2x)}{qx(1-q)^2}$$
$$= \frac{f(qx) - f(q^2x)}{qx(1-q)}$$
$$= D_q f(qx)$$
$$= D_q H_q f(x).$$

But:

$$D_q X^2 D_q f(x) = xD_q\left(x^2 \frac{f(x)-f(qx)}{x(1-q)}\right)$$
$$= \frac{1}{1-q} D_q \left(xf(x) - xf(qx)\right)$$
$$= \frac{1}{1-q}\left(\frac{xf(x)-xf(qx)}{x(1-q)} - \frac{xqf(qx)-xqf(q^2x)}{x(1-q)}\right)$$
$$-\frac{1}{(1-q)^2}\left(f(x)-(1+q)f(qx)+qf(q^2x)\right)$$

Similarly, we get:

$$X^2 D_q^2 f(x) = x^2 D_q\left(\frac{f(x)-f(qx)}{x(1-q)}\right)$$
$$= \frac{x^2}{1-q}\left(\frac{\frac{qf(x)-qf(qx)}{qx} - \frac{f(qx)+f(q^2x)}{qx}}{x(1-q)}\right)$$
$$= \frac{1}{q(1-q)}\left(qf(x)-(1+q)f(qx)+f(q^2x)\right)$$

Which gives:

$$\left[M_{0,q}, M_{2,q}\right] = \frac{1}{q(1-q)^2}\left((1+q)(-q+1)f(qx)+\left(q^2-1\right)f\left(q^2x\right)\right)$$
$$= x(1+q)\left(\frac{f(qx)-f(q^2x)}{qx}\right)$$
$$= x[2]qD_q f(qx)$$
$$= [2]qXD_q H_q f(qx)$$

We have:

$$\left[M_{1,q}, M_{2,q}\right] f(x) = \left[XD_q, X^2 D_q\right]$$
$$= XD_q X^2 D_q - X^2 D_q XD_q$$

$$XD_q X^2 D_q f(x) = xD_q\left(\frac{xf(x)-xf(qx)}{(1-q)}\right)$$
$$= \frac{x}{1-q}\left(\frac{xf(x)-xf(qx)-qxf(qx)+qxf(q^2x)}{x(1-q)}\right)$$
$$= \frac{x}{(1-q)^2}\left(f(x)-(1+q)f(qx)+qf(q^2x)\right)$$

Similarly, we have:

$$X^2 D_q XD_q f(x) = x^2 D_q\left(\frac{f(x)-f(qx)}{(1-q)}\right)$$
$$= \frac{x^2}{1-q}\left(\frac{f(x)-f(qx)-f(qx)-f(q^2x)}{x(1-q)}\right)$$
$$= \frac{x}{q(1-q)^2}\left(f(x)-2f(qx)+f(q^2x)\right)$$

Then, we get:

$$\left[M_{1,q}, M_{2,q}\right] f(x) = \frac{x}{q(1-q)^2}\left((1-q)f(qx)-(q-1)f(q^2x)\right)$$
$$= \frac{x}{(1-q)}\left(f(qx)-f(q^2x)\right)$$
$$= qx^2\left(\frac{f(qx)-f(q^2x)}{qx(1-q)}\right)$$
$$= qx^2 D_q f(qx)$$
$$= qX^2 D_q H_q f(x).$$

Proposition 3.2

For $q \in (0, 1)$ we have:

i) $[M_{0,q}, H_q] = 0$.
ii) $[M_{1,q}, H_q] = (1-q)M_{1,q}H_q$.
iii) $[M_{2,q}, H_q] = (1-q)[2]_q M_{2,q}H_q$.

Proof

We have:

$$\left[D_q, H_q\right] f(x) = D_q H_q f(x) - H_q D_q f(x)$$
$$= D_q f(qx) - H_q\left(\frac{f(x)-f(qx)}{x(1-q)}\right)$$
$$= \frac{f(qx)-f(q^2x)}{qx(1-q)} - \frac{f(qx)-f(q^2x)}{qx(1-q)}$$
$$= 0.$$

Then, we get:

$$\left[M_{0,q}, H_q\right] = 0.$$

We have:

$$\left[XD_q, H_q\right] f(x) = XD_q H_q f(x) - H_q XD_q f(x)$$
$$= xD_q f(qx) - H_q\left(xD_q f(x)\right)$$
$$= xD_q f(qx) - qxD_q f(qx)$$
$$= (1-q) XD_q H_q f(x).$$

Then, we get:

$$\left[M_{0,q}, H_q\right] = (1-q) M_{1,q} H_q.$$

We have:

$$\begin{aligned}
&\left[X^2 D_q, H_q\right] f(x) = X^2 D_q H_q f(x) - H_q\left(X^2 D_q f(x)\right) \\
&= x^2 D_q f(qx) - (qx)^2 D_q f(x) \\
&= \left(1-q^2\right) X^2 D_q H_q f(x) \\
&= (1-q)[2]_q X^2 D_q H_q f(x).
\end{aligned}$$

Then, we obtain:

$$\left[M_{2,q}, H_q\right] = (1-q)[2]_q M_{2,q} H_q.$$

which complete the proof.

Now, we give the representation theorem of the q-Riccati algebra.

Theorem 3.3

Let φ: $R_q \to gl(\mathfrak{H}^2(\mathfrak{D}_q, \mu_q))$ a linear mapping such that:

$$\begin{aligned}
\varphi(A) &= M_{0,q} \\
\varphi(B) &= M_{1,q} \\
\varphi(C) &= M_{2,q} \\
\varphi(D) &= H_q.
\end{aligned}$$

Then, $(\mathfrak{H}^2(\mathfrak{D}_q; \mu_q), \varphi)$ is a representation of R_q.

Proof

The proof follows from Proposition 3.1 and Proposition 3.2.

Author's Contributions

All authors equally contributed in this work.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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