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S-Numbers of Weighted Shift Operators on P-Summable Formal Entire Functions of M-Variables

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Abstract: The idea of multiplying a formal Taylor power series by z to make a right shift operator on the space of all square summable sequences of real numbers was due to A.L. Shield. In this work, we consider Taylor power series in m-variables and we give upper and lower estimations of s-numbers for multiplication of m- right weighted shift operators. This allowed us to estimate upper bounds for s-numbers of infinite series of m-right weighted shift operators and give some applications.

Keywords: S-Numbers, Shift Operators, Formal Power Series

Introduction

For any bounded linear operator T from a Banach space E into a Banach space F there are associated some decreasing sequences of non negative numbers called s-numbers satisfying certain conditions. Examples of s-numbers are approximation, Berrnstien, Gelfand, Kolmogorov and Tichomirov numbers. Hilbert Schmidt operators are those operators whose sequences of approximation numbers are square summable (Pietsch, 1980). Compact operators are those operators whose sequence of Kolmogorov numbers converges to zero. For more details about these and other s-numbers we refer the reader to (Pietsch, 1980; 1987). Shields (1974) gave representation for weighted shift operators as formal power series in unilateral shifts and formal Laurent series in bilateral shifts. He also suggested to express functions belonging to the space $H^2(\beta)$ by $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ with $\|f\|^2 = \sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2 \beta^2(n) < \infty$ where $\{\beta(n)\}$ is a sequence of positive numbers with $\beta(0) = 1$ and $\left\{\hat{f}(n)\right\}$ is a sequence of real numbers. In this case, to see that $\|z_k\| = \beta(k)$ he considered the following: $\hat{f}_k(n) = \delta_{nk}$, so $\|f_k(z)\|^2 = \sum_{n=0}^{\infty}\left|\hat{f}_k(n)\right|^2 \|z^n\|^2 = \sum_{n=0}^{\infty}\left|\delta_{nk}\right|^2 \|z^n\|^2 = \|z^k\|^2 \beta^2(k)$; it is clear that $\{f_k\}$ is an orthogonal basis. Hedayatian (2004) defined H_β^p to be the space of all functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ equipped with the norm $\|f\| = \left(\sum_{n=0}^{\infty}|a_n|^p \beta^p(n)\right)^{\frac{1}{p}} < \infty$ which is a generalization of the space l^p of all absolutely p summable sequences.

In this work, we consider the space $H_{\beta(I)}^p$, $1\le p<\infty$ of formal power series $f(z^I) = \sum_I a_I z^I = \sum_{\substack{i_k=0 \\ k=1,2,\cdots,m}}^{\infty} a_{(i_1,i_2,\cdots,i_m)} z_1^{i_1} z_2^{i_2}\cdots z_m^{i_m}$ in m-variables equipped with $\|f\| = \left(\sum_I |a_I|^p \beta^p(I)\right)^{\frac{1}{p}} <\infty$, where (a_I): $\underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{m-times}\to\mathbb{R}$ is a family of real numbers indexed by a multi-index I and $\beta(I) = \beta_1(i_1)\,\beta_2(i_2)\cdots\beta_m(i_m)$, where $\beta_j(i_j)$, $i_j\in\mathbb{N}^* = \{0, 1,\cdots\}$, $j = 1, 2,\cdots,m$ are positive weights such that $\beta_j(0) = 1$ for all j. The family $\left\{z^I, I\in\mathbb{N}^*\right\}$ forms an orthogonal basis to the space $H_{\beta(I)}^p$. We give upper bounds for s-numbers of a series $\sum_I c_J R^J$ (where $R^J = \prod_{k=1}^{m} R_k^{j_k}$ and $J = (j_1, j_2,\cdots, j_m)$ is an index set of m natural numbers) of unilateral weighted shift operators.

Basic Definitions and Lemmas

Definition 2.1 (Pietsch, 1980)

By $L(E, F)$, we denote the space of all bounded linear operators from a normed space E into a normed space F. A map s which assigns to every operator $T\in L(E, F)$ a

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unique sequence $\left\{s_n(T)\right\}_{n=0}^{\infty}$ is called an s-function if the following conditions are satisfied:

1. $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for $T \in L(E, F)$
2. $s_n(U+V) \leq s_n(U) + \|V\|$ for $U, V \in L(E, F)$
3. $s_n(UTV) \leq \|U\|\, s_n(T)\, \|V\|$ for $V \in L(E_0, E)$, $T \in L(E, F)$ and $U \in L(F, F_0)$
4. If $T \in L(E, F)$ and rank$T<n$, then $s_n(T) = 0$
5. $s_n(I_n) = 1$, where I_n is the identity map of the space l_2^n

We call $s_n(T)$ the n-th s-number of the operator T.

Lemma 2.2 (Faried et al., 1993)

Let $\{\tau_i\}$ be a bounded family of real numbers. Then:

$$\sup_{card\xi=r+1}\inf_{i\in\xi}\tau_i = \inf_{card\,\xi=r}\sup_{i\notin\xi}\tau_i,$$

where, $card\xi$ is the number of elements of the subset of indices ξ.

Lemma 2.3 (Beckenbach and Bellman, 1971)

Let $x_{ij} \geq 0$ for $i = 0, 1, 2, \cdots, n$ and $j = 0, 1, 2, \cdots m$. If $p \geq 1$, then:

$$\left[\sum_{i=0}^{n}\left(\sum_{j=0}^{m}x_{ij}\right)^p\right]^{\frac{1}{p}} = \sum_{j=0}^{m}\left(\sum_{i=0}^{n}x_{ij}^p\right)^{\frac{1}{p}}.$$

Definition 2.4

T By R_j and S_j, we denote the unilateral forward and backward shift operators on $H_{\beta(I)}^p$ defined by:

$$R_j f\left(z^I\right) = z_j f\left(z^I\right) = \sum_{\substack{i_k=0\\k=1,2,\cdots,m}}^{\infty} a_{\left(i_1,i_2,\cdots i_j,\cdots i_m\right)} z_1^{i_1} z_2^{i_2}\cdots z_j^{i_j+1}\cdots z_m^{i_m}$$

and:

$$S_j f\left(z^I\right) = \frac{f\left(z^I\right)-f(0)}{z_j} = \sum_{\substack{i_k=0\\k=1,2,\cdots,m}}^{\infty} a_{\left(i_1,i_2,\cdots i_j,\cdots i_m\right)} z_1^{i_1} z_2^{i_2}\cdots z_j^{i_j-1}\cdots z_m^{i_m}$$

respectively.

Lemma 2.5

The powers R_j^n and S_j^n of the operators R_j and S_j are bounded on the space $H_{\beta(I)}^p$ with $\left\|R_j^n\right\| = \sup_{i_j}\frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)}$ and $\left\|S_j^n\right\| = \sup_{i_j}\frac{\beta_j\left(i_j\right)}{\beta_j\left(i_j+n\right)}$ respectively provided that the right-hand side exists.

Proof

Let $E_j = (0, 0, \cdots 1, 0, \cdots)$ (1 in the j^{th} place). For $f(z^I) \in H_{\beta(I)}^p$, we have:

$$R_j^n f\left(z^I\right) = \sum_I a_I z^{I+nE_j} = z_j^n f\left(z^I\right) = \sum_{\substack{i_k=0\\k=1,2,\cdots,m}}^{\infty} a_{\left(i_1,i_2,\cdots i_m\right)} z_1^{i_1} z_2^{i_2}\cdots z_j^{i_j+n}\cdots z_m^{i_m},$$

and hence,

$$\left\|R_j^n\right\| = \sup_{\|f(z^I)\|\neq 0}\frac{\left\|R_j^n f\left(z^I\right)\right\|}{\left\|f\left(z^I\right)\right\|}$$

$$= \sup_{\|f(z^I)\|\neq 0}\frac{\left(\sum_I |a_I|^p\, \beta_1^p\left(i_1\right)\beta_2^p\left(i_2\right)\cdots\beta_j^p\left(i_j+n\right)\cdots\beta_m^p\left(i_m\right)\right)^{\frac{1}{p}}}{\left(\sum_I |a_I|^p\, \beta_1^p\left(i_1\right)\beta_2^p\left(i_2\right)\cdots\beta_m^p\left(i_m\right)\right)^{\frac{1}{p}}}$$

$$\leq \sup_{i_j}\frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)}.$$

On the other hand:

$$\left\|R_j^n\right\| \geq \sup_{i_j}\frac{\left\|R_j^n\left(z_1^{i_1}z_2^{i_2}\cdots z_j^{i_j}\cdots z_m^{i_m}\right)\right\|}{\left\|z_1^{i_1}z_2^{i_2}\cdots z_j^{i_j}\cdots z_m^{i_m}\right\|} = \sup_{i_j}\frac{\left\|z_1^{i_1}z_2^{i_2}\cdots z_j^{i_j+n}\cdots z_m^{i_m}\right\|}{\left\|z_1^{i_1}z_2^{i_2}\cdots z_j^{i_j}\cdots z_m^{i_m}\right\|}$$

$$= \sup_{i_j}\frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)}.$$

Therefore, $\left\|R_j^n\right\| = \sup_{i_j}\frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)}$. By the same manner, we can prove that $\left\|S_j^n\right\| = \sup_{i_j}\frac{\beta_j\left(i_j\right)}{\beta_j\left(i_j+n\right)}$.

Lemma 2.6

The operators R^J and S^J are bounded on the space $H_{\beta(I)}^p$ with:

1. $\left\|R^J\right\| = \sup_I\frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)}\frac{\beta_2\left(i_2+j_2\right)}{\beta_2\left(i_2\right)}\cdots\frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)}$
2. $\left\|S^J\right\| = \sup_I\frac{\beta_1\left(i_1\right)}{\beta_1\left(i_1+j_1\right)}\frac{\beta_2\left(i_2\right)}{\beta_2\left(i_2+j_2\right)}\cdots\frac{\beta_m\left(i_m\right)}{\beta_m\left(i_m+j_m\right)}$

provided that the right-hand side exists.

Proof

The proof is similar to that of lemma 2.5.

Main Results

Theorem 3.1

The s-numbers of the unilateral forward weighted shift operator R^J have the following upper and lower estimations:

$$\sup_{\prod_{i=1}^{m} r_i \le r+1} \sup_{card\xi_i = r_i} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)} \le s_r(R^J)$$

$$\le \inf_{\prod_{i=1}^{m} r_i \le r+1} \sup_{card\xi_i = r_i+1} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

Proof

Let $\xi_1, \xi_2, \cdots \xi_m$ be a collection of finite sets of natural numbers such that card $\xi_i = r_i$ with $\prod_{i=1}^{m} r_i \le r+1$. We consider the following projections $P_{\xi_1}, P_{\xi_2}, \cdots P_{\xi_m}$ on the space $H^p_{\beta(I)}$ with $rankP_{\xi_i} = r_i$ such that $P_{\xi_i}\left(\sum_I a_I z^I\right) = \sum_{i_j\in\xi_j, i_k\in\mathbb{N}^*, k\ne j} a_I z^I$, j = 1, 2, ⋯, m and by $P_\xi = P_{\xi_1}, P_{\xi_2}, \cdots P_{\xi_m}$ where $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ then we have $P_\xi\left(\sum_I a_I z^I\right) = \sum_{I\in\xi} a_I z^I$.

Clearly $P_\xi S^J R^J P_\xi$ is the identity operator on the space $H^p_{\beta(I)}$ and by using Definition 2.1 part (3) and lemma 2.6, we get:

$$s_r(R^J) \ge \frac{1}{\|S^J\|} = \frac{1}{\sup_{I\in\xi} \frac{\beta_1(i_1)}{\beta_1(i_1+j_1)} \cdots \frac{\beta_m(i_m)}{\beta_m(i_m+j_m)}}$$

$$= \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \frac{\beta_2(i_2+j_2)}{\beta_2(i_2)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

Since this relation is true for every i = 1, 2,⋯, m with card $\xi_i = r_i$, we get:

$$s_r(R^J) \ge \sup_{card\xi_i = r_i} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \frac{\beta_2(i_2+j_2)}{\beta_2(i_2)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

Therefore for every $\prod_{i=1}^{m} r_i \le r+1$, we get:

$$s_r(R^J)$$

$$\ge \sup_{\prod_{i=1}^{m} r_i \le r+1} \sup_{card\xi_i = r_i} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \frac{\beta_2(i_2+j_2)}{\beta_2(i_2)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

On the other hand, from the definition of the s-numbers, we define a finite rank operator $P_\xi R^J$ by:

$$P_\xi R^J\left(\sum_I a_I z^I\right) = \sum_{I\in\xi} a_I z^{I+J}.$$

Since the approximation numbers are the greatest s-numbers, we get:

$$s_r(R^J) \le \alpha_r(R^J) \le \|R^J - P_\xi R^J\|$$

$$= \sup_{\|f(z^I)\|\ne 0} \frac{\|(R^J - P_\xi R^J) f(z^I)\|}{\|f(z^I)\|}$$

$$= \sup_{\|f(z^I)\|\ne 0} \frac{\left\|\sum_{I\notin\xi} a_I z^{I+J}\right\|}{\left\|\sum_I a_I z^I\right\|}$$

$$= \sup_{\|f(z^I)\|\ne 0} \frac{\left[\sum_{I\notin\xi} |a_I|^p \beta_1^p(i_1+j_1)\cdots\beta_m^p(i_m+j_m)\right]^{\frac{1}{p}}}{\left[\sum_I |a_I|^p \beta_1^p(i_1)\cdots\beta_m^p(i_m)\right]^{\frac{1}{p}}}$$

$$\le \sup_{I\notin\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)} \frac{\left[\sum_{I\notin\xi} |a_I|^p \beta_1^p(i_1)\cdots\beta_m^p(i_m)\right]^{\frac{1}{p}}}{\left[\sum_{I\notin\xi} |a_I|^p \beta_1^p(i_1)\cdots\beta_m^p(i_m)\right]^{\frac{1}{p}}}$$

$$\le \sup_{I\notin\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

Proceeding to infimum, we get:

$$s_r(R^J) \le \inf_{card\xi_i = r_i} \sup_{I\notin\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

By using lemma 2.2 we get:

$$s_r(R^J) \le \sup_{card\xi_i = r_i+1} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

Since this relation is true for every $\prod_{i=1}^{m} r_i \le r+1$, we get:

$$s_r(R^J) \le \inf_{\prod_{i=1}^{m} r_i \le r+1} \sup_{card\xi_i = r_i+1} \inf_{I\in\xi} \frac{\beta_1(i_1+j_1)}{\beta_1(i_1)} \cdots \frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}.$$

End of the proof.

The next proposition gives upper and lower bounds to the norm of the infinite series of unilateral weighted shift operators $\sum_{n=0}^{\infty} c_n R_z^n$ on the space $H^p_{\beta(I)}$ under the condition $\{c_n\}_{n=0}^{\infty} \in l_1$.

Proposition 3.2

For the infinite series of unilateral weighted shift operators $\sum_{n=0}^{\infty} c_n R_j^n$ on the space $H_{\beta(I)}^p$ and for every $\{c_n\}_{n=0}^{\infty} \in l_1$ and $1 \le p < \infty$, we get:

$$\sup_{i_j}\left(\sum_{n=0}^{\infty}|c_n|^p\left[\frac{\beta_j(i_j+n)}{\beta_j(i_j)}\right]^p\right)^{\frac{1}{p}} \le \left\|\sum_{n=0}^{\infty}c_nR_j^n\right\| \le \sup_{i_j,n}\frac{\beta_j(i_j+n)}{\beta_j(i_j)}\sum_{n=0}^{\infty}|c_n|.$$

Proof

For every $f(z^I) \in H_{\beta(I)}^p$, we get:

$$\left\|\sum_{n=0}^{\infty}c_nR_j^n\right\| = \sup_{\|f(z^I)\|\neq 0}\frac{\left\|\sum_{n=0}^{\infty}c_nR_j^n\sum_I a^Iz^I\right\|}{\|f(z^I)\|} = \sup_{\|f(z^I)\|\neq 0}\frac{\left\|\sum_{\substack{i_k=0\\k=1,2,\cdots,m}}^{\infty}\sum_{n=0}^{i_j}(c_nR_j^n)a_{I-nE_j}z^{I-nE_j}\right\|}{\|f(z^I)\|}$$

$$= \sup_{\|f(z^I)\|\neq 0}\frac{\left(\sum_I\left[\sum_{n=0}^{i_j}|c_n||a_{I-nE_j}|\beta_1(i_1)\cdots\beta_m(i_m)\right]^p\right)^{\frac{1}{p}}}{\|f(z^I)\|}$$

$$= \sup_{\|f(z^I)\|\neq 0}\frac{\left(\sum_I\left[\sum_{n=0}^{\infty}|c_J||a_I|\beta_1(i_1)\cdots\beta_j(i_j+n)\cdots\beta_m(i_m)\right]^p\right)^{\frac{1}{p}}}{\|f(z^I)\|}$$

By using Lemma 2.3, we get:

$$\left\|\sum_{n=0}^{\infty}c_nR_j^n\right\| \le \sup_{i_j,n}\frac{\beta_j(i_j+n)}{\beta_j(i_j)}\sum_{n=0}^{\infty}|c_n|.$$

On the other hand:

$$\left\|\sum_{n=0}^{\infty}c_nR_j^n\right\| \ge \frac{\left\|\left(\sum_{n=0}^{\infty}c_nR_j^n\right)z^I\right\|}{\|z^I\|} =$$

$$= \frac{\left\|\sum_{n=0}^{\infty}c_nz^{I+nE_j}\right\|}{\beta_1(i_1)\cdots\beta_m(i_m)}$$

$$= \frac{\left(\sum_{n=0}^{\infty}|c_n|^p\beta_1^p(i_1)\cdots\beta_j^p(i_j+n)\cdots\beta_m^p(i_m)\right)}{\beta_1(i_1)\cdots\beta_m(i_m)}$$

$$= \left(\sum_{n=0}^{\infty}|c_n|^p\left[\frac{\beta_j(i_j+n)}{\beta_j(i_j)}\right]^p\right)^{\frac{1}{p}}$$

Therefore:

$$\left\|\sum_{n=0}^{\infty}c_nR_j^n\right\| \ge \sup_{i_j}\left(\sum_{n=0}^{\infty}|c_n|^p\left[\frac{\beta_j(i_j+n)}{\beta_j(i_j)}\right]^p\right)^{\frac{1}{p}}.$$

End of the proof.

Remark 3.3

For $p = 1$ in the previous proposition, the right and left estimations coincide to get an exact estimation.

Proposition 3.4

For the unilateral forward weighted shift operator $\sum_J c_J R^J$ on the space $H_{\beta(I)}^p$ and for every $\{c_J\}_{J=0}^{\infty}$ satisfying $\sum_J |c_J| < \infty$ and $1 \le p < \infty$, we get:

$$\sup_I\left(\sum_J|c_J|^p\left[\frac{\beta_1(i_1+j_1)}{\beta_1(i_1)}\right]^p\cdots\left[\frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}\right]^p\right)^{\frac{1}{p}}$$

$$\le \left\|\sum_J c_JR^J\right\| \le \sup_{I,J}\frac{\beta_1(i_1+j_1)}{\beta_1(i_1)}\cdots\frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}\sum_J|c_J|$$

Proof

For every $f(z^I) \in H_{\beta(I)}^p$, we get:

$$\left\|\sum_J c_JR^J\right\| = \sup_{\|f(z^I)\|\neq 0}\frac{\left\|\sum_I\sum_{J=0}^{I}c_Ja_{I-J}z^I\right\|}{\|f(z^I)\|} = \sup_{\|f(z^I)\|\neq 0}\frac{\left\|\sum_I\sum_{\substack{j_k=0\\k=1,,\cdots,m}}^{i_k}c_Ja_{I-J}z^I\right\|}{\|f(z^I)\|}$$

$$= \sup_{\|f(z^I)\|\neq 0}\frac{\left(\sum_I\left[\sum_{\substack{j_k=0\\k=1,,\cdots,m}}^{i_k}|c_J||a_{I-J}|\beta_1(i_1)\cdots\beta_m(i_m)\right]^p\right)^{\frac{1}{p}}}{\|f(z^I)\|}$$

$$= \sup_{\|f(z^I)\|\neq 0}\frac{\left(\sum_I\left[\sum_{\substack{j_k=0\\k=1,,\cdots,m}}^{\infty}|c_J||a_I|\beta_1(i_1+j_1)\cdots\beta_m(i_m+j_m)\right]^p\right)^{\frac{1}{p}}}{\|f(z^I)\|}$$

By using Lemma 2.3, we get:

$$\left\|\sum_J c_JR^J\right\| \le \sup_{I,J}\frac{\beta_1(i_1+j_1)}{\beta_1(i_1)}\cdots\frac{\beta_m(i_m+j_m)}{\beta_m(i_m)}\sum_J|c_J|.$$

On the other hand:

$$\left\|\sum_J c_JR^J\right\| \ge \frac{\left\|\left(\sum_J c_JR^J\right)z^I\right\|}{\|z^I\|}$$

$$= \frac{\left\|\sum_J c_Jz^{I+J}\right\|}{\beta_1(i_1)\cdots\beta_m(i_m)}$$

$$= \frac{\left(\sum_J |c_J|^p \beta_1^p\left(i_1+j_1\right)\beta_m^p\left(i_m+j_m\right)\right)^{\frac{1}{p}}}{\beta_1\left(i_1\right)\cdots\beta_m\left(i_m\right)}$$

$$= \left(\sum_J |c_J|^p \left[\frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)}\right]^p \cdots \left[\frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)}\right]^p\right)^{\frac{1}{p}}$$

Therefore:

$$\left\|\sum_J c_J R^J\right\| \geq \sup_I \left(\sum_J |c_J|^p \left[\frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)}\right]^p \cdots \left[\frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)}\right]^p\right)^{\frac{1}{p}}.$$

End of the proof.

In the following proposition, we get an upper estimation to the s-numbers of the unilateral forward shift operator of the form of an infinite series $\sum_{m=0}^{\infty} c_m R_z^m$ on the space $H_{\beta(I)}^p$.

Theorem 3.5

For the unilateral forward shift operator $\sum_{n=0}^{\infty} c_n R_z^n$ on the space $H_{\beta(I)}^p$, the s-numbers of this operator (such that $\{c_n\}_{n=0}^{\infty} \in l^1$) are given by:

$$s_r\left(\sum_{n=0}^{\infty} c_n R_j^n\right) \leq \sup_{i_j, card\,\xi = r+1} \inf_{n\in\xi} \frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)} \sum_{n\notin\xi} |c_n|.$$

Proof

Let ξ be a subset of the set of natural numbers $\mathbb{N}$ with card $\xi = r$:

$$s_r\left(\sum_{n=0}^{\infty} c_n R_j^n\right) \leq \left\|\sum_{n=0}^{\infty} c_n R_j^n - \sum_{n\in\xi} c_n R_j^n\right\| = \left\|\sum_{n\notin\xi} c_n R_j^n\right\|.$$

From proposition 3.2, we get $s_r\left(\sum_{n=0}^{\infty} c_n R_j^n\right) \leq \sup_{i_j, n\notin\xi} \frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)} \sum_{n\notin\xi} |c_n|.$

Since this relation is true for every set ξ with card $\xi = r$ and by using lemma 2.2, we get:

$$s_r\left(\sum_{n=0}^{\infty} c_n R_j^n\right) \leq \sup_{i_j, card\,\xi = r+1} \inf_{n\in\xi} \frac{\beta_j\left(i_j+n\right)}{\beta_j\left(i_j\right)} \sum_{n\notin\xi} |c_n|.$$

This completes the proof.

Theorem 3.6

For the unilateral forward weighted shift operator $\sum_J c_J R^J$ such that $\sum_J c_J < \infty$ on the space $H_{\beta(I)}^p$, the s-numbers for this operator satisfy:

$$s_r\left(\sum_J c_J R^J\right)$$
$$\leq \inf_{\prod_{i=1}^m r_i \leq r} \sup_{I, card\,\xi_i = r_i+1} \inf_{J\in\xi} \frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)} \cdots \frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)} \sum_{J\notin\xi} |c_J|.$$

Proof

Let ξ_i, $i = 1, 2, \cdots, m$ be m subsets of the set of natural numbers $\mathbb{N}$ with card $\xi_i = r_i$ such that $\prod_{i=1}^m r_i \leq r$. From proposition 3.4, we get:

$$s_r\left(\sum_J c_J R^J\right) \leq \left\|\sum_J c_J R^J - \sum_{J\in\xi} c_J R^J\right\| = \left\|\sum_{J\notin\xi} c_J R^J\right\|$$
$$\leq \sup_{I, J\notin\xi} \frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)} \cdots \frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)} \sum_{J\notin\xi} |c_J|.$$

Since this relation is true for every set ξ_i with card $\xi_i = r_i$ and by using lemma 2.2, we get:

$$s_r\left(\sum_J c_J R^J\right) \leq \sup_{I, card\,\xi_i = r_i+1} \inf_{J\in\xi} \frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)} \cdots \frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)} \sum_{J\notin\xi} |c_J|.$$

Since this relation is true for every $\prod_{i=1}^m r_i \leq r$, we get:

$$s_r\left(\sum_J c_J R^J\right)$$
$$\leq \inf_{\prod_{i=1}^m r_i \leq r} \sup_{I, card\,\xi_i = r_i+1} \inf_{J\in\xi} \frac{\beta_1\left(i_1+j_1\right)}{\beta_1\left(i_1\right)} \cdots \frac{\beta_m\left(i_m+j_m\right)}{\beta_m\left(i_m\right)} \sum_{J\notin\xi} |c_J|.$$

This completes the proof.

Some Applications on S-numbers of Some Operators

In this section, we use the formal expansion of some well known functions of more than one variable and consider it as series of forward weighted shift operators. For these series, we give upper estimations of its s-numbers.

By using the power series of three entire functions $g_1(x)\sum_{n=0}^{\infty} a_n x^n, g_2(y) = \sum_{n=0}^{\infty} b_n y^n$ and $g_3(y) = \sum_{n=0}^{\infty} c_n z^n$, we define shift operators $R_{g_1} f(x, y, z) = g_1(x) f(x, y, z)$, $R_{g_2} f(x, y, z) = g_2(y) f(x, y, z)$ and $R_{g_3} f(x, y, z) = g_3(z) f(x, y, z)$ on the space $H_{\beta(i_1, i_2, i_3)}^p$.

Lemma 4.1

Let $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ be three infinite series then we have:

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m} a_k b_{m-k} c_{n-m}$$

Proof:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n \sum_{n=0}^{\infty} c_n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \sum_{n=0}^{\infty} c_n \\ &= \sum_{n=0}^{\infty} A_n \sum_{n=0}^{\infty} c_n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_m c_{n-m} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m} a_k b_{m-k} c_{n-m}\end{aligned}$$

Example 4.2

Considering the function $f(x, y, z) = e^{x^2+3y}$ as a series of shift operators on the space $H^p_{\beta(i_1,i_2)}$, we have the following expansion:

$$\begin{aligned}e^{x^2+3y} &= R_{e^{x^2}} R_{e^{3y}} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{x^{2k}}{k!}\right)\left(\frac{3^{m-k}}{(m-k)!}y^{m-k}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{3^{m-k}}{k!(m-k)!}x^{2k}y^{m-k}.\end{aligned}$$

We get:

$$\begin{aligned}s_r\left(e^{x^2+3y}\right) &= s_r\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{3^{m-k}}{k!(m-k)!}x^{2k}y^{m-k}\right) \\ &\leq \inf_{r_1 r_2 \leq r}\ \sup_{I,\, card\, \xi_i = r_i + 1}\ \inf_{J\in\xi}\frac{\beta_1(i_1+2k)}{\beta_1(i_1)}\frac{\beta_2(i_2+m-k)}{\beta_2(i_2)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{3^{m-k}}{k!(m-k)!}\end{aligned}$$

where, $I = (i_1, i_2)$ and $J = (2k, m\text{-}k)$.

Example 4.3

Considering the function $f(x, y, z) = \sin x^2 \sin 5y \sin \frac{z}{3}$ as a series of shift operators on the space $H^p_{\beta(i_1,i_2,i_3)}$, we have the following expansion:

$$\sin x^2 \sin 5y \sin\frac{z}{3}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^n 5^{2(m-k)}}{3^{2(n-m)+1}(2k+1)!(2m-2k+1)!(2n-2m+1)!}x^{4k+2}y^{2m-2k+1}z^{2n-2m+1}$$

we get:

$$\begin{aligned}s_r\left(\sin x^2 \sin 5y \sin\frac{z}{3}\right) &= s_r\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^n 5^{2(m-k)}}{3^{2(n-m)+1}(2k+1)!(2m-2k+1)!(2n-2m+1)!}x^{4k+2}y^{2m-2k+1}z^{2n-2m+1}\right) \\ &\leq \inf_{\prod_{i=1}^{3} r_i \leq r}\ \sup_{I,\, card\, \xi_i = r_i + 1}\ \inf_{J\in\xi}\frac{\beta_1(i_1+4k+2)}{\beta_1(i_1)}\frac{\beta_2(i_2+2m-2k+1)}{\beta_2(i_2)}\frac{\beta_3(i_3+2n-2m+1)}{\beta_3(i_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^n 5^{2(m-k)}}{3^{2(n-m)+1}(2k+1)!(2m-2k+1)!(2n-2m+1)!},\end{aligned}$$

where, $I = (i_1, i_2, i_3)$ and $J = (4k + 2, 2m\text{-}2k + 1, 2n\text{-}2m + 1)$.

Example 4.4

Considering the function $f(x, y, z) = \sin x \cos ye^z$ as a series of shift operators on the space $H^p_{\beta(i_1,i_2,i_3)}$, we have the following expansion:

$$\sin x \cos ye^z = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^m}{(2k+1)!(2m-2k)!(n-m)!}x^{2k+1}y^{2m-2k}z^{n-m}.$$

we get:

$$s_r\left(\sin x\cos ye^z\right) = s_r\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^m}{(2k+1)!(2m-2k)!(n-m)!}x^{2k+1}y^{2m-2k}z^{n-m}\right)$$

$$\leq \inf_{\prod_{i=1}^{3} r_i \leq r}\ \sup_{I, card\,\xi_i = r_i+1}\ \inf_{J\in\xi}\frac{\beta_1(i_1+2k+1)}{\beta_1(i_1)}\frac{\beta_2(i_2+2m-2k)}{\beta_2(i_2)}\frac{\beta_3(i_3+n-2)}{\beta_3(i_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{(-1)^m}{(2k+1)!(2m-2k)!(n-m)!}x^{2k+1}y^{2m-2k}z^{n-m}.$$

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Author's Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Ethics

The authors declare that there is no conflict of interests regarding the puplication of this article.

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