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The Burr X Nadarajah Haghighi Distribution: Statistical Properties and Application to the Exceedances of Flood Peaks Data

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Abstract: In this study, new version from the Nadarajah Haghighi model is proposed. The introduced model has a failure rate function that may change in different directions. Statistical properties of the new density function are derived along with the analysis of variance, Skewness and Kurtosis. Parameter estimates are obtained by the method of maximum likelihood. Illustration of real data set was employed to measure flexibility of the new model. A simulation study was executed to test performance of the proposed version.

Keywords: Burr X Family, Nadarajah Haghighi Model, Modeling, Failure Rate Function

Introduction

Recently, a new generalization of the Exponential (E) distribution as an alternative model to the gamma (Ga), Weibull (W) and Exponentiated-Exponential (EE) distributions was proposed by Nadarajah and Haghighi (2011). The Cumulative Distribution Function (CDF) of Nadarajah and Haghighi (NH) model the is given by:

$$G_{\alpha,\lambda}(x) = 1 - \exp\left[1 - (\lambda x + 1)^{\alpha}\right]$$

and the corresponding Probability Density Function (PDF) is:

$$g_{\alpha,\lambda}(x) = \alpha\lambda(\lambda x + 1)^{\alpha-1}\exp\left[1 - (\lambda x + 1)^{\alpha}\right],$$

where, α and λ are the shape and scale parameters, respectively, which are both greater than zero. Clearly, when $\alpha = 1$, we have the standard Exponential (E) model. Nadarajah and Haghighi (2011) pointed out that the density function ($g_{\alpha\lambda}(x)$) has the attractive feature of always having zero mode. They also showed that larger values of α in ($g_{\alpha\lambda}(x)$) will lead to faster decay of the upper tail. In this study, we will refer to the proposed distribution as the Burr X Nadarajah Haghighi (BXNH) model. According to Yousof *et al*. (2017a), the CDF and the PDF of the BX-G family of distributions can be expressed as:

$$F_{\theta,\underline{\xi}}(x) = 2\theta\int_0^{G_{\underline{\xi}}(x)/\bar{G}_{\underline{\xi}}(x)} t\exp\left(-t^2\right)\left[1-\exp\left(-t^2\right)\right]^{\theta-1}dt = \left\{1-\exp\left[-\left(G_{\underline{\xi}}(x)/\bar{G}_{\underline{\xi}}(x)\right)^2\right]\right\}^{\theta}, \quad (1)$$

and:

$$f_{\theta,\underline{\xi}}(x) = 2\theta\frac{g_{\underline{\xi}}(x)G_{\underline{\xi}}(x)}{\bar{G}_{\underline{\xi}}(x)^3}\exp\left[-\left(G_{\underline{\xi}}(x)/\bar{G}_{\underline{\xi}}(x)\right)^2\right] \times\left\{1-\exp\left[-\left(G_{\underline{\xi}}(x)/\bar{G}_{\underline{\xi}}(x)\right)^2\right]\right\}^{\theta-1}, \quad (2)$$

respectively, where $\theta > 0$ is the shape parameter, $g_{\underline{\xi}}(x)$ and $G_{\underline{\xi}}(x)$ denote the PDF and the CDF of the baseline model with parameter vector $\underline{\xi}$. To this end, we use $G_{\alpha,\lambda}(x)$, $G_{\alpha,\lambda}(x)$ and (1) to obtain the four-parameter BXNH PDF (for $x > 0$) as:

$$F(x) = F_{\theta,\alpha,\lambda}(x) = \left[1-\exp\left(-\left\{\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left[1-(\lambda x+1)^{\alpha}\right]}\right\}^2\right)\right]^{\theta}, \quad (3)$$

with corresponding PDF:

$$\begin{aligned} f(x) = f_{\theta,\alpha,\lambda}(x) &= 2\theta\alpha\lambda(\lambda x+1)^{\alpha-1} \\ &\times\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left\{2\left[1-(\lambda x+1)^{\alpha}\right]\right\}} \\ &\times\exp\left(-\left\{\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left[1-(\lambda x+1)^{\alpha}\right]}\right\}^2\right) \\ &\times\left[1-\exp\left(-\left\{\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left[1-(\lambda x+1)^{\alpha}\right]}\right\}^2\right)\right]^{\theta-1} \end{aligned} \quad (4)$$

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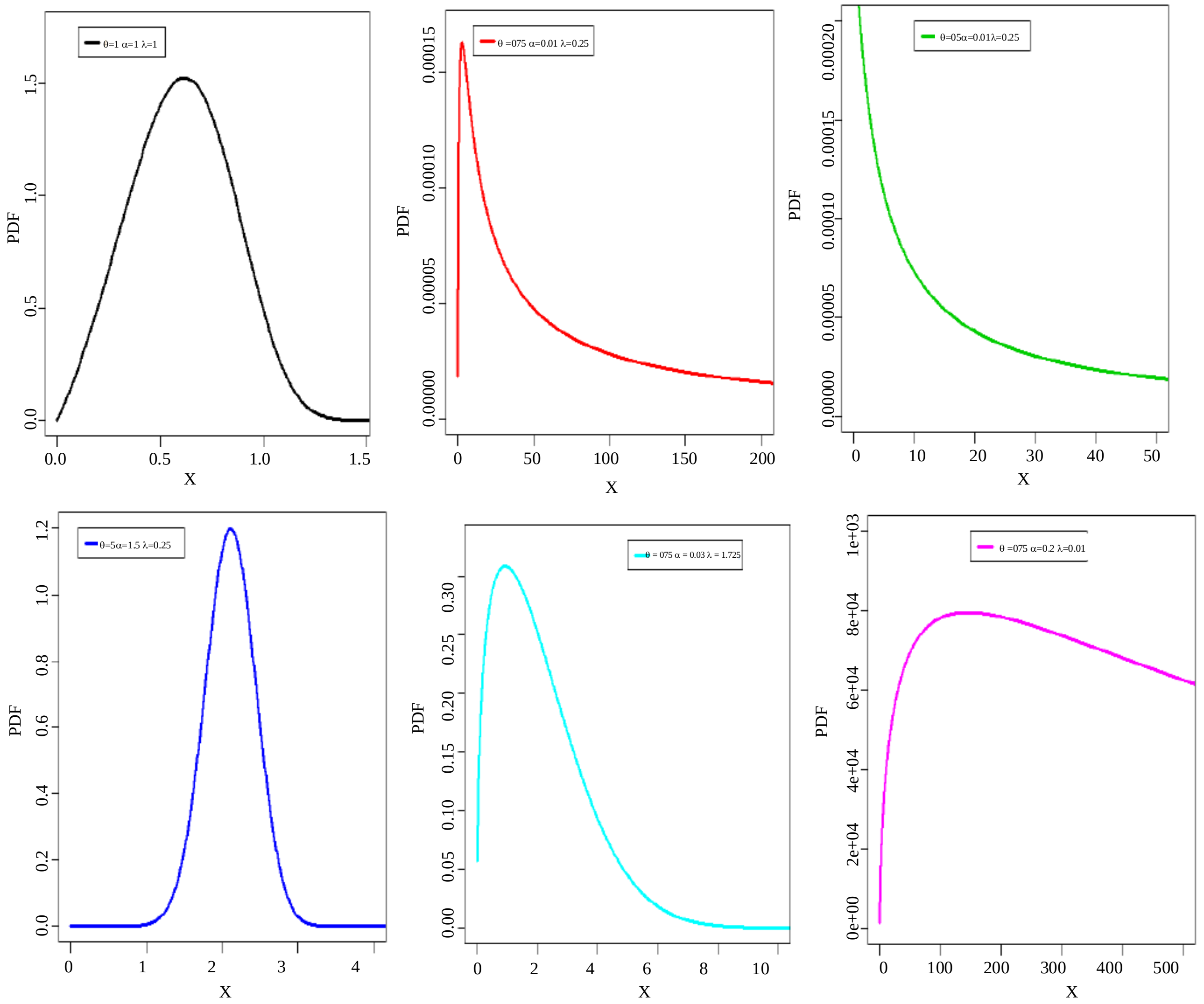
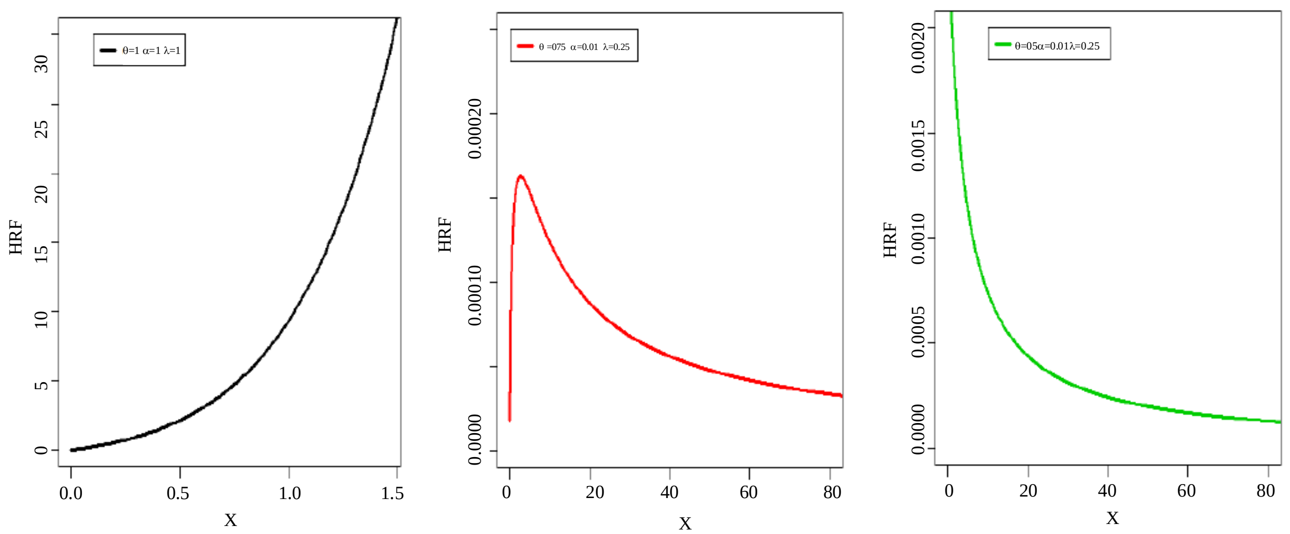


Fig. 1: Plots of the BXNH PDF at some parameters value



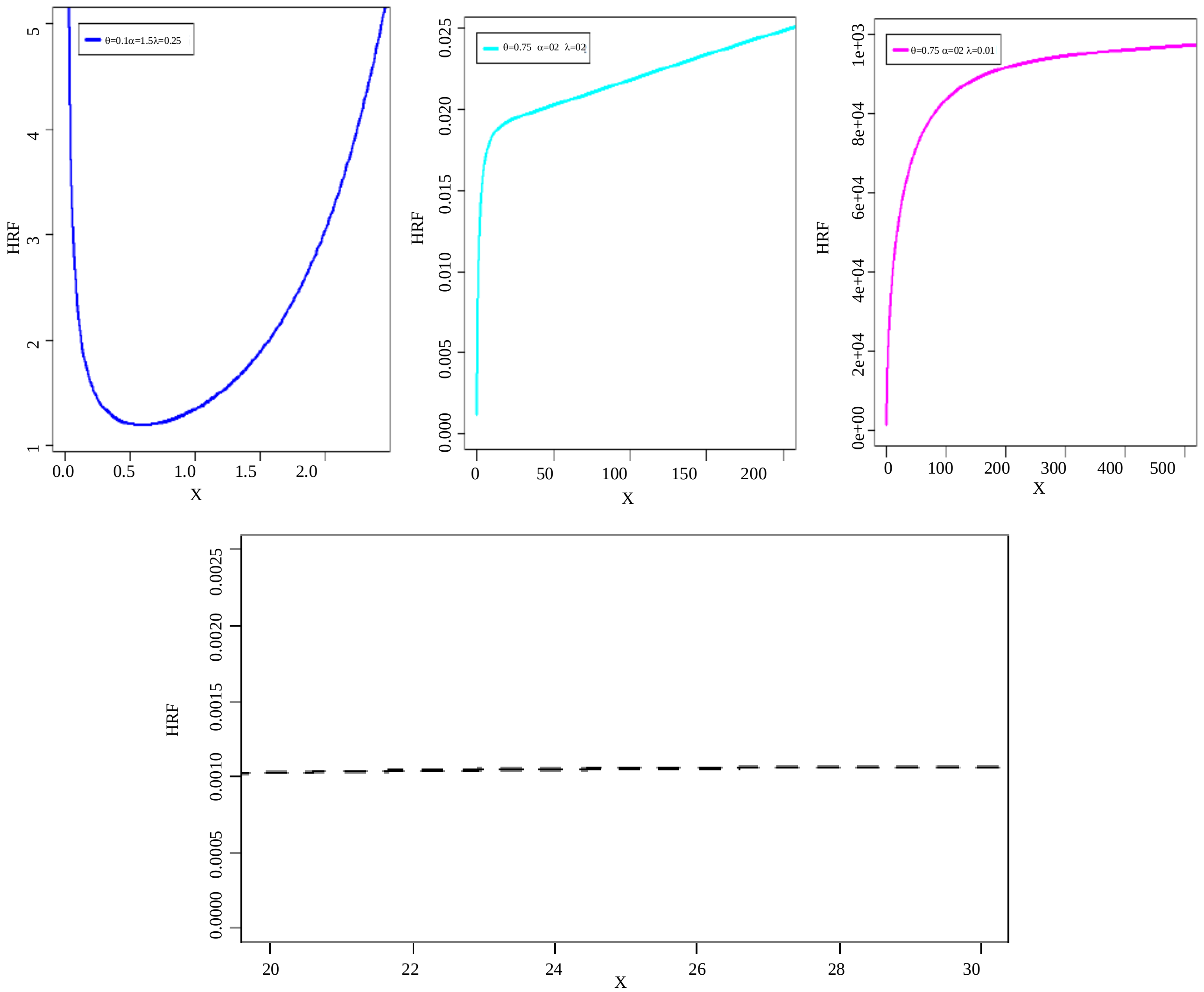


Fig. 2: Plots of the BXNH HRF at some parameters value

The Reliability Function (RF), Hazard Rate Function (HRF), Reversed Hazard Rate Function (RHRF) and Cumulative Hazard Rate Function (CHRF) of X can be derived with the well-known relationships. For θ = 1, we have the Rayleigh NH (RNH) model. For θ = 1, we have the Rayleigh Exponential (RE) model. For θ = 1, we have the BX Exponential (BXE) model. Figure 1 shows that the new density function can take a unimodal, symmetric and right skewed shapes. Figure 2 shows that the HRF may be increasing or upside-down or decreasing or bathtub (U) or increasing then constant or constant shaped failure rate function.

Useful Representation

In this section, we provide a useful simple representation for the BXNH density function. Consider the following power series:

$$(1-\Upsilon)^b \big|_{(|\Upsilon|<1 and\, b>0)} = \sum_{\varsigma=0}^{\infty} \Upsilon^{\varsigma} \frac{(-1)^{\varsigma}\Gamma(1+b)}{\varsigma!\Gamma(1+b-\varsigma)}. \tag{5}$$

Applying (5) to (4) we have:

$$\begin{aligned} f(x) &= 2\theta\alpha\lambda(\lambda x+1)^{\alpha-1} \\ &\times \frac{\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}}{\exp\left\{2\left[1-(\lambda x+1)^{\alpha}\right]\right\}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\theta)}{m!\Gamma(\theta-m)} \\ &\times \exp\left[-(1+m)\left\{\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left[1-(\lambda x+1)^{\alpha}\right]}\right\}^2\right] \end{aligned} \tag{6}$$

Applying the power series to the term:

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$$\exp\left[-(1+m)\left\{\frac{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]}{\exp\left[1-(\lambda x+1)^{\alpha}\right]}\right\}^{2}\right],$$

Then, $f(x)$ in (6) becomes:

$$f(x)=2\theta\alpha\lambda(\lambda x+1)^{\alpha-1}\exp\left[1-(\lambda x+1)^{\alpha}\right] \times\sum_{m,\omega=0}^{\infty}\frac{(-1)^{m+\omega}(1+m)^{\omega}\Gamma(\theta)}{m!\omega!\Gamma(\theta-m)} \times\frac{\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\omega+1}}{\left\{\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\omega+3}}. \quad (7)$$

Consider the series expansion:

$$(1-\Upsilon)^{-b}\,|_{(|\Upsilon|<1\,and\,b>0)}=\sum_{q=0}^{\infty}\Upsilon^{q}\frac{\Gamma(b+q)}{q!\Gamma(b)}. \quad (8)$$

Applying the expansion in (8) to (7) for the term $\{\exp[1-(\lambda x+1)^{\alpha}\}^{2\omega+3}$, $f(x)$ in (7) becomes:

$$f(x)=2\theta\sum_{m,\omega,k=0}^{\infty}\frac{\Gamma(\theta)\,\Gamma(2\omega+\kappa+3)[2\omega+\kappa+2]}{m!\omega!\kappa!\Gamma(\theta-m)\Gamma(2\omega+3)[2\omega+\kappa+2]} \times(-1)^{m+\omega}(1+m)^{\omega}\alpha\lambda(\lambda x+1)^{\alpha-1}\exp\left[1-(\lambda x+1)^{\alpha}\right] \times\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\omega+\kappa+1}.$$

This can be written as:

$$f(x)=\sum_{\omega,\kappa=0}^{\infty}v_{\omega,\kappa}\pi_{2\omega+\kappa+2}(x,\alpha,\lambda), \quad (9)$$

where:

$$v_{\omega,\kappa}=\frac{2\theta(-1)^{\omega}\Gamma(\theta)\Gamma(2\omega+\kappa+3)}{\omega!\kappa!\Gamma(2\omega+3)(2\omega+\kappa+2)}\sum_{m=0}^{\infty}\frac{(-1)^{m}(1+m)^{\omega}}{m!\Gamma(\theta-m)},$$

and:

$$\pi_{2\omega+\kappa+2}(x,\alpha,\lambda)=(2\omega+\kappa+2)\alpha\lambda(\lambda x+1)^{\alpha-1} \times\exp\left[1-(\lambda x+1)^{\alpha}\right] \times\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\omega+\kappa+1}.$$

Equation (9) show that the density of X can be expressed as a linear mixture representation of Exponentiated NH (ENH) density. So, several mathematical properties of the BXNH model can be obtained by knowing those of the ENH distribution. Similarly, the CDF of the BXNH model can also be expressed as:

$$F(x)=\sum_{\omega,\kappa=0}^{\infty}v_{\omega,\kappa}\Pi_{2\omega+\kappa+2}(x,\alpha,\lambda) \quad (10)$$

where:

$$\Pi_{2\omega+\kappa+2}(x,\alpha,\lambda)=\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\omega+\kappa+2}$$

Is the CDF of the ENH density with power parameter $(2\omega+\kappa+2)$.

Mathematical and Statistical Properties

Moments and Moment Generating Function

The r^{th} ordinary moment of X is given by:

$$\mu'_{\Upsilon}=E(X^{r})=\int_{-\infty}^{\infty}f(x)x^{r}dx.$$

Then we obtain:

$$\mu'_{r}=\sum_{\omega,\kappa,\zeta=0}^{\infty}\sum_{i=0}^{r}v_{\omega,\kappa}c_{\zeta,i}^{(2\omega+\kappa+2,r)}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right), \quad (11)$$

where:

$$c_{\zeta,i}^{(\alpha,r)}=\frac{a}{\lambda^{r}}\frac{(-1)^{r+\zeta-i}\exp(\zeta+1)}{(\zeta+1)^{i\alpha^{-1}+1}}\binom{-1+a}{\zeta}\binom{r}{i}$$

and:

$$\mu'_{r}=\sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{2\omega+\kappa+1}\sum_{i=0}^{r}v_{\omega,\kappa}c_{\zeta,i}^{(2\omega+\kappa+2,r)} \times\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)|_{(2\omega+\kappa+2>0\,and\,integer)}.$$

Setting $r = 1$ in (11), we have the mean of X:

$$\mu'_{1}=\sum_{\omega,\kappa,\zeta=0}^{\infty}\sum_{i=0}^{1}v_{\omega,\kappa}c_{\zeta,i}^{(2\omega+\kappa+2,1)}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right),$$

where:

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$$c_{\zeta,i}^{(a,1)} = \frac{a}{\lambda}\frac{(-1)^{\Upsilon+\zeta-i}\exp(\zeta+1)}{(\zeta+1)^{i\alpha^{-1}+1}}\binom{-1+a}{\zeta}\binom{1}{i}$$

and:

$$\mu_r' = \sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{2\omega+\kappa+1}\sum_{i=0}^{1} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,1)} \times\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)|_{(2\omega+\kappa+2>0 \text{ and integer})},$$

where:

$$\Gamma(a,v) = \int_v^{\infty} z^{-1+a}\exp(-z)dz$$

denotes the complementary incomplete gamma function, which can be evaluated in MATHEMATICA, R, etc. The variance Var(X), skewness Ske(X) and kurtosis Ku(X) measures can be calculated from the ordinary moments using well-known relationships (see subsection 3.7).The variance (Var(X)), skewness (Ske(X)) and kurtosis (Ku(X)) can also be calculated from the ordinary moments using well-known relationships. Here, we provide a formulae for the moment generating function (MGF) $M_X(t) = \mathbf{E}(e^{tX})$ of X. Clearly, the MGF can be derived from (9) as:

$$M_X(t) = \sum_{\omega,\kappa,\zeta,r=0}^{\infty}\sum_{i=0}^{r}\frac{t^r}{r!} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,r)}\Gamma\left(i\alpha^{-1}+1,\ \zeta+1\right),$$

and:

$$M_X(t) = \sum_{\omega,\kappa,r=0}^{\infty}\sum_{\zeta=0}^{2\omega+\kappa+1}\sum_{i=0}^{r}\frac{t^r}{r!} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,r)} \times\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)|_{(2\omega+\kappa+2>0 \text{ and integer})}.$$

Incomplete Moments

The s^{th} incomplete moment, say $\varphi_s(t)$, of X can be expressed from (9) as:

$$\varphi_s(t) = \int_{-\infty}^{t} f(x)x^s dx = \sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{\infty}\sum_{i=0}^{s} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,s)} \times\left[\begin{matrix}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)\\ -\Gamma\left(i\alpha^{-1}+1,(\zeta+1)(1+\lambda t)^{\alpha}\right)\end{matrix}\right], \quad (12)$$

And:

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{2\omega+\kappa+1}\sum_{i=0}^{s} v_{\omega,\kappa}\ c_{\zeta,i}^{(2\omega+\kappa+2,s)} \times\left[\begin{matrix}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)\\ -\Gamma\left(i\alpha^{-1}+1,(\zeta+1)(1+\lambda t)^{\alpha}\right)\end{matrix}\right]|_{(2\omega+\kappa+2>0 \text{ and integer})}.$$

The mean deviations about the mean $[\tau_1 = \mathbf{E}(|X-\mu_1'|)]$ and about the median $[\tau_2 = \mathbf{E}(|X-\mathbf{M}|)]$ of X are given by:

$$\tau_1 = -2\varphi_1(\mu_1') + 2\mu_1' F(\mu_1') \text{ and } \tau_2 = -2\varphi_1(M) + \mu_1',$$

respectively, where $\mu'_1 = \boldsymbol{E}(X)$, $M = Q\left(\frac{1}{2}\right) = Median(X)$ is the median, $F(\mu_1')$ is easily calculated from (3) and $\varphi_1(t)$ is the first incomplete moment given by (12) with s = 1. The $\varphi_1(t)$ can be derived from (12) as:

$$\varphi_1(t) = \sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{\infty}\sum_{i=0}^{1} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,1)} \times\left[\begin{matrix}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)\\ -\Gamma\left(i\alpha^{-1}+1,(\zeta+1)(1+\lambda t)^{\alpha}\right)\end{matrix}\right], \quad (1)$$

and:

$$\varphi_1(t) = \sum_{\omega,\kappa=0}^{\infty}\sum_{\zeta=0}^{2\omega+\kappa+1}\sum_{i=0}^{1} v_{\omega,\kappa} c_{\zeta,i}^{(2\omega+\kappa+2,1)} \times\left[\begin{matrix}\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)\\ -\Gamma\left(i\alpha^{-1}+1,(\zeta+1)(1+\lambda t)^{\alpha}\right)\end{matrix}\right]|_{(2\omega+\kappa+2>0 \text{ and integer})}.$$

Probability Weighted Moments (PWMs)

The (s, r)th PWM of X following the Burr type X generator, say $\mu_{s,r}$, is formally defined by:

$$\mu_{s,r} = E\left\{X^S F(X)^r\right\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)dx.$$

Using equations (3), (4) and (10) we can write:

$$f(x)F(x)^r = \sum_{\omega,\kappa=0}^{\infty}\tau_{\omega,\kappa}\pi_{2\omega+\kappa+2}(x),$$

where:

$$\tau_{\omega,\kappa} = \frac{2\theta(-1)^{\varpi}\Gamma(2\omega+\kappa+3)}{\omega!\kappa!\Gamma(2\omega+3)(2\omega+\kappa+2)} \times\sum_{m=0}^{\infty}(-1)^m(1+m)^{\omega}\binom{\theta(r+1)-1}{m}.$$

Then, the (s, r)$^{\text{th}}$ PWM of X can be expressed as:

$$\mu_{s,r} = \sum_{\omega,\kappa,\varsigma=0}^{\infty}\sum_{i=0}^{r}\tau_{\omega,\kappa}c_{\zeta,i}^{(2\omega+\kappa+2,s)}\Gamma\left(i\alpha^{-1}+1,\varsigma+1\right),$$

and also:

$$\mu_{s,r} = \sum_{\omega,\kappa=0}^{\infty}\sum_{\xi=0}^{2\omega+\kappa+1}\sum_{i=0}^{r}\tau_{\omega,\kappa}c_{\zeta,i}^{(2\omega+\kappa+2,s)} \times\Gamma\left(i\alpha^{-1}+1,\zeta+1\right)|_{(2\omega+\kappa+2>0 \text{and integer})}.$$

Residual and Reversed Residual Life

The n^{th} moment of the residual life, say:

$$a_n(t) = \boldsymbol{E}\left[(X-t)^n\right]|_{X>t \text{ and } n=1,2...},$$

which uniquely determine the $F(x)$. The n^{th} moment of the residual life of X is given by:

$$a_n(t) = \frac{\int_t^\infty (x-t)^n dF(x)}{1-F(t)}.$$

Therefore:

$$a_n(t) = \frac{1}{1-F(t)} \sum_{\omega,\kappa,\varsigma=0}^{\infty} \sum_{i=0}^{n} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r} \times v_{\omega,\kappa} \boldsymbol{c}_{\zeta,i}^{(2\omega+\kappa+2,n)} \Gamma\left(i\alpha^{-1}+1, \zeta+1\right),$$

or:

$$a_n(t) = \frac{1}{1-F(t)} \sum_{\omega,\kappa=0}^{\infty} \sum_{\zeta=0}^{2\omega+\kappa+1} \sum_{i=0}^{n} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r} \times \boldsymbol{v}_{\omega,\kappa} \boldsymbol{c}_{\zeta,i}^{(2\omega+\kappa+2,n)} \Gamma\left(i\alpha^{-1}+1, \zeta+1\right)|_{(2\omega+\kappa+>0 \text{ and integer})}.$$

The Mean Residual Life (MRL) function or the life expectation at age t can be defined by:

$$a_{n=1}(t) = \boldsymbol{E}\left[(X-t) \mid X > t\right],$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in the last equation.

The n^{th} moment of the reversed residual life, say:

$$A_n(t) = \boldsymbol{E}\left[(t-X)^n\right]|_{(X\le t,\ t>0 \text{ and } n=1,2,...)},$$

then, we obtain:

$$A_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes:

$$A_n(t) = \frac{1}{F(t)} \sum_{\omega,\kappa,\zeta=0}^{\infty} \sum_{i=0}^{n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r} \boldsymbol{v}_{\omega,\kappa} \boldsymbol{c}_{\zeta,i}^{(2\omega+\kappa+2,n)} \times \left[\Gamma\left(i\alpha^{-1}+1, \zeta+1\right) - \Gamma\left(i\alpha^{-1}+1, (\zeta+1)(1+\lambda t)^\alpha\right)\right],$$

Or:

$$A_n(t) = \frac{1}{F(t)} \sum_{\omega,\kappa=0}^{\infty} \sum_{\zeta=0}^{2\omega+\kappa+1} \sum_{i=0}^{n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r} \boldsymbol{v}_{\omega,\kappa} \boldsymbol{c}_{\zeta,i}^{(2\omega+\kappa+2,n)} \times \left[\begin{array}{c} \Gamma\left(i\alpha^{-1}+1, \varsigma+1\right) \\ -\Gamma\left(i\alpha^{-1}+1, (\varsigma+1)(1+\lambda t)^\alpha\right) \end{array}\right]\Bigg|_{(2\omega+\kappa+2>0 \text{ and integer})}.$$

The Mean Waiting Time (MWT) or the Mean Inactivity Time (MIT) which also called the mean reversed residual life function, is given by:

$$A_{n=1}(t) = \boldsymbol{E}\left[(t-X)^n\right]|_{(X\le t,\ t>0 \text{ and } n=1)}$$

and it represents the waiting time elapsed since the failure of an item on condition had occurred in $(0,t)$. The MIT of the BXNH distribution can be obtained easily by setting $n = 1$ in the above equation of $A_n(t)$.

Stress-Strength Reliability Model

Stress-strength reliability model is the most widely approach used for reliability estimation. The stress-strength reliability model is used in many applications of physics and engineering such as system collapse and strength failure. In stress strength reliability modeling, $\mathbf{R}_{X2<X1} = Pr(X_2<X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1.

The system only fails when the applied stress exceeds its strength. This means that component will be satisfied for $X_1>X_2$. Hence the performance of a system can be considered as $\mathbf{R}_{X2<X1}$ and naturally arise in electrical and electronic systems. The reliability, $\mathbf{R}_{X2<X1}$, can also be explained as the probability that the system is strong enough to defeat the stress imposed on it.

Let X_1 and X_2 be two independent rvs with BXNH$(\theta_1, \alpha, \lambda)$ and BXNH$(\theta_2, \alpha, \lambda)$ distributions, respectively. The PDF of X_1 and the CDF of X_2 can be written from Equations (9) and (10), respectively, as:

$$f_1(x;\theta_1,\alpha,\lambda) = 2\theta_1 \sum_{i,\varsigma,\kappa=0}^{\infty} \frac{(-1)^{i+\varsigma}(i+1)^\varsigma \Gamma(\theta_1)\Gamma(2\omega+\kappa+3)}{i!\zeta!k!\Gamma(\theta_1-i)\Gamma(2\omega+3)} \times \alpha\lambda(\lambda x+1)^{\alpha-1} \exp\left[1-(\lambda x+1)^\alpha\right] \times \left\{1-\exp\left[1-(\lambda x+1)^\alpha\right]\right\}^{2i+\kappa+1}$$

and:

$$F_2(x;\theta_2,\alpha,\lambda) = 2\theta_2 \sum_{h,\omega,m=0}^{\infty} \frac{\Gamma(\theta_2)\Gamma(2\omega+m+3)}{\Gamma(\theta_2-h)\Gamma(2\omega+3)(2\omega+m+2)} \times \frac{(-1)^{h+\omega}(h+1)^\omega}{h!\omega!m!} \left\{1-\exp\left[1-(\lambda x+1)^\alpha\right]\right\}^{2\omega+m+2}$$

Then, the reliability is defined by:

$$\boldsymbol{R}_{X_2<X_1}=\int_0^\infty f_1\left(x;\,\theta_1,\alpha,\lambda\right)F_2\,(x;\theta_2,\alpha,\lambda)dx.$$

We can write:

$$\boldsymbol{R}_{X_2<X_1}=\sum_{\zeta,\kappa,\omega,m=0}^{\infty}q_{\zeta,\kappa,\omega,m}\int_0^\infty\pi_{2\zeta+2\omega+\kappa+m+4}(x)dx,$$

Where:

$$q_{\zeta,\kappa,\omega,m}=4\theta_1\theta_2\sum_{\zeta,\kappa,\omega,m=0}^{\infty}\frac{(-1)^{\zeta+\omega}\Gamma(2\zeta+\kappa+3)(2\omega+m+3)}{\zeta!\kappa!\omega!m!\Gamma(2\zeta+3)\Gamma(2\omega+3)}$$
$$\times\sum_{i,h=0}^{\infty}\frac{(-1)^{i+h}(i+1)^{\zeta}(h+1)^{\omega}}{(2\omega+m+2)(2\zeta+\kappa+2\omega+m+4)}$$
$$\times\binom{\theta_1-1}{i}\binom{\theta_2-1}{h},$$

and:

$$\pi_{2\zeta+2\omega+\kappa+m+4}(x)=(2\zeta+\kappa+2\omega+m+4)$$
$$\times\alpha\lambda(\lambda x+1)^{\alpha-1}\exp\left[1-(\lambda x+1)^{\alpha}\right]$$
$$\times\left\{1-\exp\left[1-(\lambda x+1)^{\alpha}\right]\right\}^{2\zeta+\kappa+2\omega+m+3}.$$

Thus, the reliability, $\boldsymbol{R}_{X_2<X_1}$, can be expressed as:

$$\boldsymbol{R}_{X_2<X_1}=\sum_{\zeta,\kappa,\omega,m=0}^{\infty}q_{\zeta,\kappa,\omega,m}.$$

Order Statistics

Let $X_{1,\ldots,}X_n$ be a Random Sample (RS) from the BXNH of distribution and let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. The PDF of i^{th} order statistic, say $X_{i:n}$, can be written as:

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F^{j+i-1}(x),\qquad(13)$$

where, $B(.,.)$ is the beta function. Using (3), (4), (9) and (10) in equation (13) we get:

$$f(x)F(x)^{j+i-1}=\sum_{\omega,\kappa=0}^{\infty}\Upsilon_{\omega,\kappa}\pi_{2\omega+\kappa+2}(x,\alpha,\lambda),$$

Where:

$$\Upsilon_{\omega,\kappa}=\frac{2\theta(-1)^{\omega}\Gamma(2\omega+\kappa+3)}{\omega!\kappa!\Gamma(2\omega+3)(2\omega+\kappa+2)}$$
$$\times\sum_{m=0}^{\infty}(-1)^m(1+m)^{\omega}\binom{\theta(j+i)-1}{m}.$$

The PDF of $X_{i:n}$ can be expressed as:

$$f_{i;n}(x)=\sum_{\omega,\kappa=0}^{\infty}\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-i}{j}\Upsilon_{\omega,\kappa}}{B(i,n-i+1)}\pi_{2\omega+\kappa+2}(x,\alpha,\lambda).\qquad(14)$$

Then, the density function of the BXNH order statistics is a mixture of ENH density. Based on (14), the moments of $X_{i:n}$ can be expressed as:

$$\boldsymbol{E}\left(X_{i:n}^{q}\right)=\sum_{\omega,\kappa,h=0}^{\infty}\sum_{d=0}^{r}\sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}\Upsilon_{\omega,\kappa}}{B(i,n-i+1)}$$
$$\times\boldsymbol{c}_{h,d}^{(2\omega+\kappa+2,q)}\Gamma\left(\frac{d}{\alpha}+1,1+h\right).$$

Numerical Analysis for the *E*(*X*); Var(*X*), Ske(*X*) and Ku(*X*) Measures

Numerical analysis for the *E*(*X*), Var(*X*), Ske(*X*) and Ku(*X*) are calculated in Table 1 using (10) and well-known relationships for some selected values of parameter θ,α and λ using the R software. Based on Table 1 we note that:

1. The skewness of the BXNH distribution is always positive
2. The kurtosis of the BXNH distribution can be only more than 3
3. The parameter λ has a xed e⁄ect on the Ske(*X*) and Ku(*X*) for all di⁄erent values of and *:* When = 5 and = 0*:*25*; Ske*(*X*) = 0*:*7646761 and *Ku*(*X*) = 3*:*892269 for any value of the parameter λ . when θ = 2 and α = 0*:*15*; Ske*(*X*) = 1*:*799314 and *Ku*(*X*) = 8*:*140326 for any value of the parameter λ .
4. The mean of the BXNH distribution increases as θ increases
5. The mean of the proposed model decreases as α and λ increases

Maximum Likelihood Estimation

Let $x_{1,\ldots,}x_n$ be a rs from BXNH distribution with parameter vector $\underline{\Psi}=(\theta,\alpha,\lambda)^{\tau}$. The log-likelihood function for, $\underline{\Psi}$ say $\ell(\underline{\Psi})$, is given by:

Table 1: E(X), Var(X), Ske(X) and Ku(X) of the BXNH distribution

θ	α	λ	E(X)	Var(X)	Ske(X)	Ku(X)
0.001	0.25	0.5	0.0289654	0.3563506	34.5865	1566.503
0.01			0.2859682	3.474523	10.92097	158.4655
0.1			2.544613	27.62755	3.412509	17.8054
0.5			8.812294	68.26591	1.511783	5.728996
1			13.12034	79.13151	1.110063	4.475175
2			17.95289	80.92004	0.8832963	4.014145
5			24.49028	74.37878	0.7646761	3.892269
10			29.2648	67.32041	0.749148	3.926701
0.5	0.1	0.5	304.8507	433498	5.533706	55.54326
	0.15		41.42335	3283.812	2.899656	15.58616
	0.2		15.8246	295.9851	1.980476	8.291567
	0.25		8.812294	68.26591	1.511783	5.728996
	0.3		5.872409	24.87189	1.226266	4.523618
	0.35		4.32819	11.73789	1.033561	3.85533
	0.4		3.397696	6.514068	0.8945249	3.443576
	0.45		2.783351	4.031865	0.7893849	3.170498
5	0.25	0.1	122.4514	1859.47	0.7646761	3.892269
		0.5	24.49028	74.37878	0.7646761	3.892269
		1	12.24514	18.5947	0.7646761	3.892269
		5	2.449028	0.7437878	0.7646761	3.892269
		20	0.6122569	0.04648674	0.7646761	3.892269
		50	0.2449028	0.007437878	0.7646761	3.892269
2	0.15	1	50.34097	1536.089	1.799314	8.140326
		10	5.034097	15.36089	1.799314	8.140325
		30	1.678032	1.706766	1.799314	8.140325
		100	0.5034097	0.153609	1.799313	8.140325

$$\begin{aligned}\ell(\underline{\Psi}) &= n\log 2 + n\log\theta + n\log\alpha + n\log\lambda \\ &+ (\alpha-1)\sum_{i=0}^{n}\log\left(1+\lambda x_i\right) + \sum_{i=0}^{n}\log\left\{1-\exp\left[1-\left(1+\lambda x_i\right)^{\alpha}\right]\right\} \\ &+ (\theta-1)\sum_{i=0}^{n}\log\left[1-\exp\left(-\left\{\frac{1-\exp\left[1-\left(1+\lambda x_i\right)^{\alpha}\right]}{\exp\left[1-\left(1+\lambda x_i\right)^{\alpha}\right]}\right\}^{2}\right)\right] \\ &- 2\sum_{i=0}^{n}\left[1-\left(1+\lambda x_i\right)^{\alpha}\right] - \sum_{i=0}^{n}\left\{\frac{1-\exp\left[1-\left(1+\lambda x_i\right)\right]}{\exp\left[1-\left(1+\lambda x_i\right)^{\alpha}\right]}\right\}^{2}.\end{aligned}$$

$\ell(\underline{\Psi})$ can be maximized either by using the deferent programs like R (optima function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating (14). The score vector elements, $\mathbf{U}(\underline{\Psi}) = \left(\frac{\partial l(\underline{\Psi})}{\partial\theta}, \frac{\partial l(\underline{\Psi})}{\partial\alpha}, \frac{\partial l(\underline{\Psi})}{\partial\lambda}\right)^{T}$ are easily to be derived.

Simulation Studies

In this section, we simulate the BXNH model by taking *n* = 20, 50, 150, 500 and 1000. For each sample size (*n*), we evaluate the ML Estimations (MLEs) of the parameters. Then, we repeat the process 1000 times (i.e. *N* = 1000) and compute the averages of the estimates (AEs) and the Mean Squared Errors (MSEs). Table 2 gives all numerical results of the simulation experiments.

The numerical results in Table 2 indicate that the MSEs and the bias of θ, α and λ decay towards zero when *n* increases for all settings of θ, α and λ as expected under the asymptotic theory or large sample theory. The AEs of the parameters tend to be closer to the true parameter values I: $\theta = 2.5$, $\alpha = 1.5$ and $\lambda = 2.0$; **II**: $\theta = 1.5$, $\alpha = 2.5$ and $\lambda = 1.5$ when *n* increases. These results support that the asymptotic normal model provides good approximation to the finite sample model of the MLEs.

Data Analysis

In this section, we present an application based on the real data set to show the flexibility of the BXNH distribution. First, we compare BXNH with the RNH, the odd Lindley NH distribution (OLNH) (Yousof *et al*., 2017b), Proportional Reversed Hazard Rate (PRHRNH) (new), exponentiated Weibull NH (New), the Gamma-NH (GNH) (Ortega *et al*., 2015), Marshall-Olin NH (MONH) (Lemont *et al*., 2016), exponentiated NH (ENH) (Lemonte, 2013), beta-NH (BNH) (Dias *et al*., 2018), the standard NH distributions. Other useful extension of the NH model such as the Topps-Leone NH distribution (Yousof *et al*., 2017b) and extended exponentiated NH model (Alizadeh *et al*., 2018).

Table 2: AEs and MSE for N = 1000

N	Θ	AE	MSE	Θ	AE	MSE
	I			II		
20	θ	2.864044	0.2047603	θ	1.5872651	0.2913098
	α	1.3611143	0.8063241	α	2.8140401	0.3550603
	λ	1.7934323	0.1981093	λ	1.8054763	0.6151902
50	θ	2.6351883	0.1895662	θ	1.5767954	0.2024421
	α	1.4522802	0.3918004	α	2.7051811	0.2628198
	λ	1.8702466	0.1210564	λ	1.7271091	0.4545328
150	θ	2.5603383	0.1119021	θ	1.5652243	0.1299292
	α	1.4922332	0.0931274	α	2.6124758	0.2028198
	λ	1.9609656	0.1054192	λ	1.6099811	0.3013233
500	θ	2.5244465	0.0551823	θ	1.5003211	0.0913652
	α	1.5004343	0.0576872	α	2.5105512	0.0832017
	λ	1.9745479	0.0305103	λ	1.5311971	0.1023321
1000	θ	2.5003231	0.0004291	θ	1.5000112	0.0011432
	α	1.5004411	0.0060651	α	2.5005491	0.0065762
	λ	2.0041123	0.0012018	λ	1.5003243	0.0055492

Table 3: Estimates of the competitive models fitted to the Choulakian and Stephens data

Model		Estimates (SD)		
NH (α, λ)	0.841	0.1094		
	(0.259)	(0.059)		
RNH (θ, α, λ)	0.125	6.28		
	(0.012)	(2.919)		
BXNH (θ, α, λ)	0.446	0.232	0.408	
	(0.147)	(0.087)	(0.478)	
OLNH (θ, α, λ)	0.7293	0.2519	1.8065	
	(0.6059)	(0.052)	(3.355)	
PRHRNH (θ, α, λ)	0.364	1.714	0.031	
	(0.068)	(1.191)	(0.031)	
GNH (a, α, λ)	0.7286	1.9299	0.0242	
	(0.1385)	(1.7591)	(0.0312)	
MONH (a, α, λ)	23.77	0.0011	0.2660	
	(5.5053)	(0.0003)	(0.0895)	
ENH (a, α, λ)	0.7289	1.7126	0.0309	
	(0.1404)	(1.2607)	(0.0330)	
BNH (a, b, α, λ)	0.8381	316.0285	0.6396	0.0003
	(0.1215)	(4.2194)	(0.8227)	(0.0004)
EWNH (θ, a, α, λ)	2.7591	0.3989	0.4732	0.6129
	(1.742)	(0.167)	(0.158)	(0.959)

The model selection is applied using the estimated log-likelihood$\left(\ell\left(\underline{\psi}\right)\right)$, Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AI_C), Consistent Akaike information criteria (CAI_C), Bayesian information criterion (BI_C), and Hannan-Quinn information criterion (HQI_C). AI_C, CAI_C, BI_C and HQI_C:

$$AI_C = -2\ell\left(\underline{\psi}\right) + 2n_{(p)},$$

$$CAI_C = -2\ell\left(\underline{\psi}\right) + \frac{2n_{(p)},}{n-\left[n_{(p)}\right]-1},$$

$$BI_C = -2\ell\left(\underline{\psi}\right) + p\log\left[n_{(p)}\right],$$

$$HQI_C = -2\ell\left(\underline{\psi}\right) + 2\left[n_{(p)}\right]\log\left(\log n\right),$$

where, $n_{(p)}$ is the number of the estimated model parameters and n is sample size. In general, the smaller values of AI_C, CAI_C, BI_C, HQI_C and K-S indicate to the better t to the data set and the biggest log-likelihood and p values of p values of the K-S statistics is chosen. Second, Total Time on Test (TTT) plot is given for the used data set. Finally, we present the estimated PDF, estimated CDF, estimated HRF, P-P and Kaplan-Meier survival plots of the BXNH for the used data set (the exceedances of flood peaks data).

The used data corresponds to the exceedances of food peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. These data consist of 72 exceedances for the years 1958 1984, rounded to one decimal place (see Choulakian and Stephens (2001)).

Table 4: Statistics of the competitive models fitted to the Choulakian and Stephens data

Model	Loglike	AIc	CAIc	BIc	HQIc	K-S(p-value)
BXNH(θ, α, λ)	-250.438	506.88	507.23	513.71	509.6	0.0980 (-0.50)
RNH(α, λ)	-251.722	507.44	507.62	513.99	509.7	0.10629 (0.3901)
NH(α, λ)	-251.9874	507.97	508.15	512.53	509.79	0.12444 (-0.2148)
OLNH(α,α, λ)	-250.589	507.18	507.53	514.01	509.9	0.1009 (-0.4565)
PRHRNH(θ, α, λ)	-300.83	607.66	608.02	614.49	610.38	0.24985 (0.00025)
GNH(α, α, λ)	-250.917	507.834	508.187	514.66	510.55	0.1065 (-0.388)
MONH(α, α, λ)	- 51.087	508.175	508.53	515.005	510.894	0.1074 (-0.3771)
EWNH(θ,α, α, λ)	-250.032	508.064	508.66	517.17	511.69	0.0974 (-0.5)
ENH(α, α, λ)	-250.925	507.849	508.202	514.679	510.57	0.1067 (-0.3859)
BNH(α,b, α, λ)	-251.356	510.713	511.31	519.82	514.34	0.1044 (-0.4127)

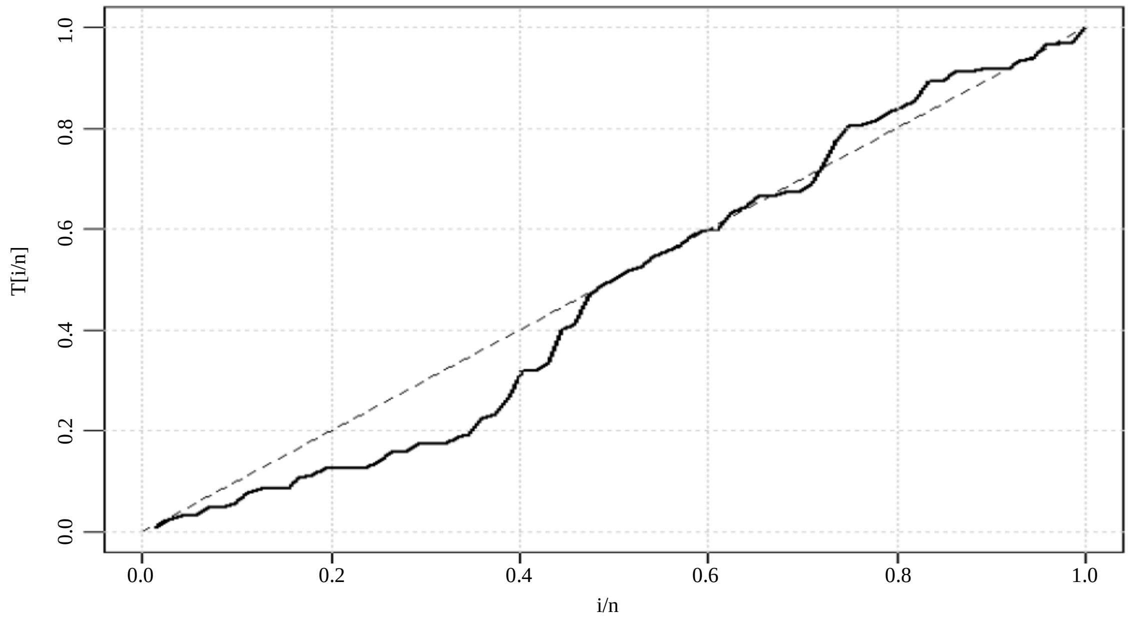
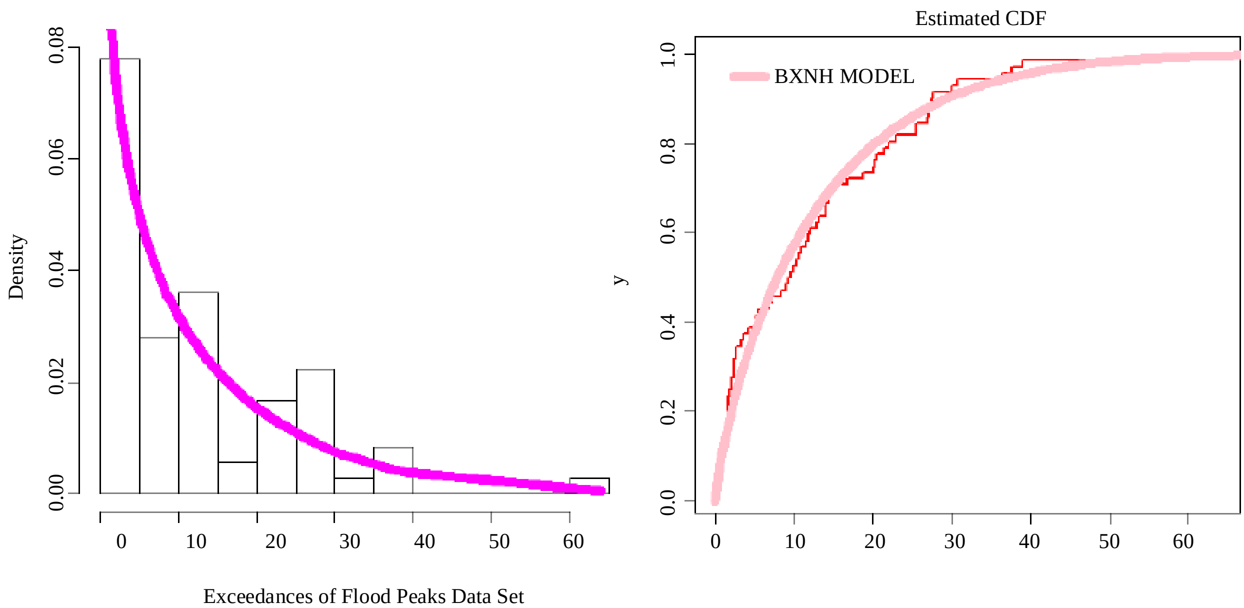


Fig. 3: TTT plot of the exceedances of flood peaks data



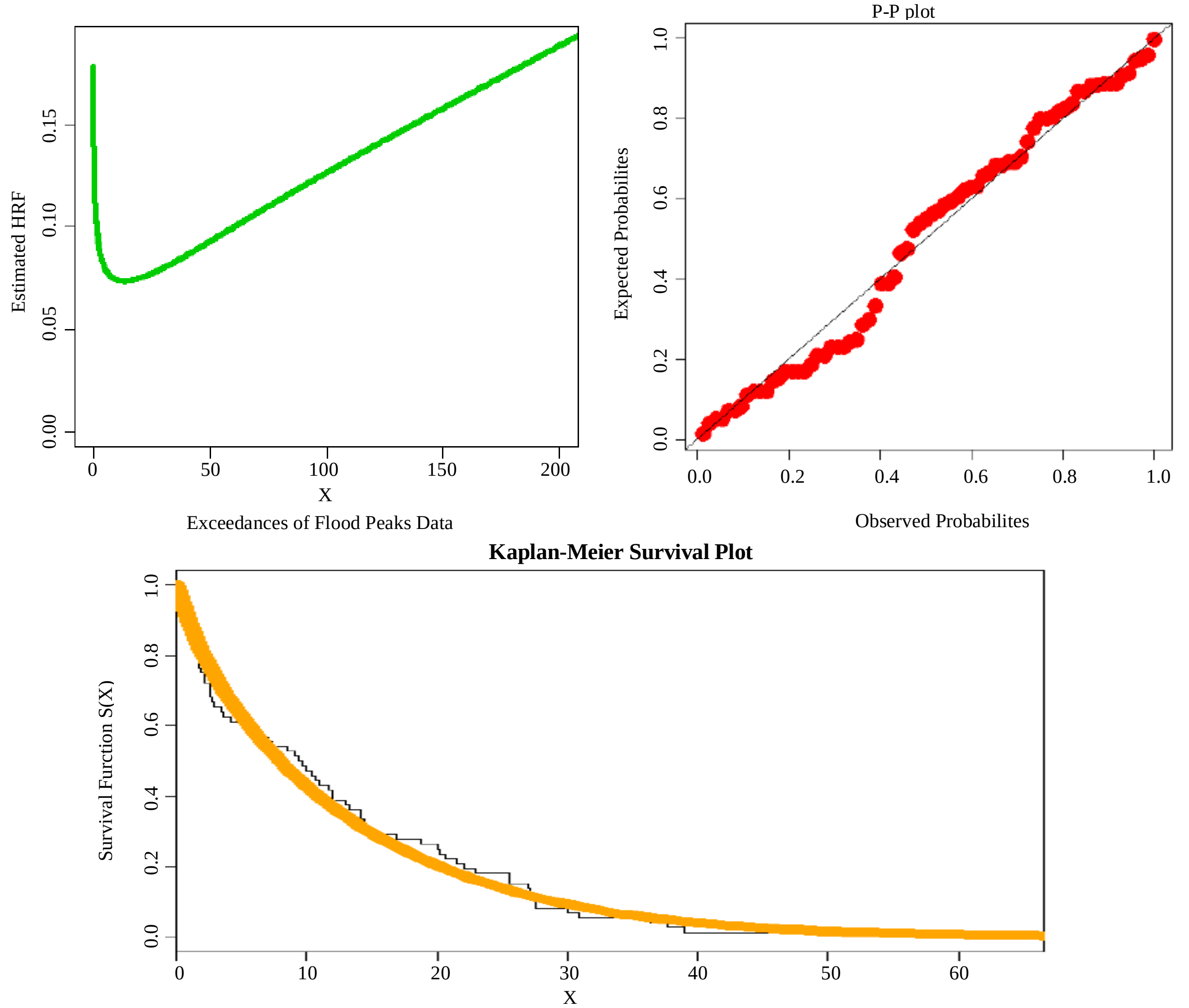


Fig. 4: Estimated PDF, CDF, HRF, P-P, Kaplan-Meier survival plots of the BXNH for the exceedances of flood peaks data

This data also have been applied by Lemonte (2013) for the ENH distribution. In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, an important tool called the TTT plot (see Aarset (1987)) is useful. The TTT plot for the exceedances of flood peaks data in Fig. 3 denotes that the failure rate function of these data is a bathtub-shaped (**U**) function.

All results of this application are listed in Table 3 and 4. These results show that the OLNH distribution has the lowest values for AI_C, CAI_C, BI_C, HQI_C and K-S values and also has the biggest estimated log-likelihood and p-value for the K-S statistics among all the fitted models. Thus, it could be chosen as the best model under these criteria and compared to the other fitted models.

Based on the estimated values of parameters given in Table 3 we note that the $\mathbf{E}(X) = 12{:}03718;\ Var(X) = 155.4608$, $Ske(X) = 1.741001$ and $Ku(X) = 6.801245$. Finally, we plot estimated functions for the density, CDF, P-P, Kaplan-Meier survival plots of the BXNH for the exceedances of flood peaks data in Fig. 4. Clearly, the BXNH distribution provides a closer fit to the empirical PDF and CDF. Also, from these figures, we get a bathtub-shaped (**U**-shaped) for the estimated HRF for the exceedances of flood peaks data, which is coincide with the TTT plot given is Fig. 3.

Conclusion

In this article, a new three-parameter version of the Nadarajah Highlight (NH) model is introduced and studied. The new density can be expressed as a straight-forward linear mixture of exponentiated Nadarajah Haghighi (ENH) density. It was shown that failure rate function of the new model can be increasing, upside-down, decreasing, bathtub, increasing then constant and constant. Some of its statistical properties including the ordinary moments, incomplete moments, moment generating function, probability weighted moments, order statistics, moment of residual life and reversed residual life have been derived. Measures of variance, skewness and kurtosis were given by a numerical analysis. A Monte Carlo simulation study is conducted to assess the performance of the maximum likelihood method. The flexibility of the new model is illustrated by a real data set. We hope that the new distribution attract wider applications in areas such as economics (income inequality), survival and lifetime data analysis, hydrology, engineering, meteorology and others.

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Author’s Contributions

The authors jointly developed and written the paper.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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