Editorials

The Rough Intuitionistic Fuzzy Zweier Lacunary Ideal Convergence of Triple Sequence Spaces

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Abstract: We introduced and studied the concept of *I*-convergence of triple sequences in metric spaces where *I* is an ideal. The concept of *I*-convergence has a wide application in the field of number theory, trigonometric series, summability theory, probability theory, optimization and approximation theory. In this article, we introduce rough intuitionistic fuzzy Lacunary ideal convergent of triple sequence spaces via zwier operators. We discuss general topological properties.

Keywords: Rough Intuitionistic Fuzzy Metric Space, *I*-convergence, I_σ-Cauchy, Zweier Triple Sequence Spaces

Introduction

The idea of rough convergence was first introduced by Phu (2001; 2002; 2003) in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree *r*.

Aytar (2008a) studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar (2008b) studied that the *r*-limit set of the sequence is equal to intersection of these sets and that *r*-core of the sequence is equal to the union of these sets. Dündar and Cakan (2014) investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of *I*-convergence of a triple sequence spaces which is based on the structure of the ideal *I* of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let *K* be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m, n, k) \in K: m \le i, n \le j, k \le \ell\}$. Then the natural density of *K* is given by:

$$\delta(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where, $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

A triple sequence (real or complex) can be defined as a function x: $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner *et al*. (2007; Sahiner and Tripathy, 2008) Esi (2014; Esi and Necdet Catalbas, 2014; Esi and Savas, 2015; Esi *et al*., 2016; 2018), Dutta *et al*. (2013), Subramanian and Esi (2015), Debnath *et al*. (2015) and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where:

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \,(m,n,k=1,2,3...).$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if:

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if:

$$| x_{mnk} |^{\frac{1}{m+n+k}} \to 0 \, as \, m,n,k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric:

$$d(x,y) = \sup_{m,n,k} \left\{ | x_{mnk} - y_{mnk} |^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \quad (1.1)$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in Γ^3. Let $\phi = \{finite \, sequences\}$.

Let λ and μ be two sequence spaces and $A = \left(a_{k\ell}^{mn}\right)$ is an infinite four dimensional matrix of real or complex numbers $a_{k\ell}^{mn}$, where $k, \ell, m, n \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ and we do not it by writing $A: \lambda \to \mu$.

If for every sequence $x = (x_{mnk}) \in \lambda$ the sequence $Ax = \{A(Ax)_{mn}\} \in \mu$, then A transform of $x \in \mu$, where $(Ax)_{mnk} = \sum_m \sum_n \sum_k a_{k\ell}^{mn} x_{mnk}, (k,\ell \in \mathbb{N})$.

By $(\lambda: \mu)$, we denote the class of matrices A such that $A: \lambda \to \mu$. Thus $a \in (\lambda, \mu)$ if and only if $(Ax) \in \mu$ for every $x \in \lambda$.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method.

The Z^p-transformation of the sequence $x = (x_{mnk})$, (i.e.,) $y_{ijk} = px_{ijk} + (1-p) x_{i-1j-1k-1}$. Z^p denotes the matrix $Z^p = \left(Z_{mnk}^{ijk}\right)$ defined by:

$$Z_{mnk}^{ijk} = \begin{cases} p, & if \, (mnk - ijk); \\ 1-p, & if \, (i-1=m, j-1=n, k-1=k); \\ 0, otherwise \end{cases}$$

The Zweier sequence space Z_{Γ^3} as follows:

$$Z_{\Gamma^3} = \left\{ x = (x_{mnk}) \in \omega^3 : Z^p x \in \Gamma^3 \right\}$$

In this study, we are working on to connect to studies which was mentioned above in order to introduce rough intuitionistic triple sequence spaces and study some algebraic and topological properties on these spaces.

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$; which is normal fuzzy convex, upper semi-continuous and the X^0 is bounded where $X^0 := cl\{x \in \mathbb{R}^3 : X(x) > 0\}$

and cl is the closure operator. These properties imply that for each $\alpha \in (0, 1]$, the α-level set X^α defined by:

$$X^\alpha = \left\{ x \in \mathbb{R}^3 : X(x) \geq \alpha \right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha \right]$$

is a non empty compact convex subset of $\mathbb{R}^3$.

The supremum metric d on the set $L(\mathbb{R}^3)$ is defined by:

$$d(X,Y) = \sup_{\alpha \in [0,1]} \max \left(| \underline{X}^\alpha - \underline{Y}^\alpha |, | \overline{X}^\alpha - \overline{Y}^\alpha | \right).$$

Now, given $X, Y \in L(\mathbb{R}^3)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0,]$.

We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L(\mathbb{R}^3)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu'$ for all upper bounds μ'.

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L(\mathbb{R}^3)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence spaces (X_{mnk}) is defined by:

$$\lim_{mnk \to \infty} \inf X_{mnk} := \inf A_X.$$
$$\lim_{mnk \to \infty} \sup X_{mnk} := \sup B_X.$$

Where:

$A_X := \{\mu \in L(\mathbb{R}^3) : \text{The set } \{(m, n, k) \in \mathbb{N}^3 : X_{mnk} < \mu\} \text{ is infinite}\}$

$B_X := \{\mu \in L(\mathbb{R}^3) : \text{The set } \{(m, n, k) \in \mathbb{N}^3 : X_{mnk} < \mu\} \text{ is infinite}\}$

Now, given two fuzzy numbers $X, Y \in L(\mathbb{R}^3)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0, 1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L(\mathbb{R}^3)$, which is defined by:

$$a_1(x) = \begin{cases} 1, & if \, x = a, \\ 0, & otherwise \end{cases}.$$

An order interval in $L(\mathbb{R}^3)$ is defined by $[X, Y] :=$ $\{Z \in L(\mathbb{R}^3) : X \le Z \le Y\}$, where $X, Y \in L(\mathbb{R}^3)$.

A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0, 1]$ and $\mu_1, \mu_2 \in E$.

Definitions and Preliminaries

In this section, we present some preliminary definitions and results related to intuitionistic fuzzy metric triple sequence spaces and that will be used throughout the article.

Definition 2.1

The five-tuple $(X, \mu, v, *, \lozenge)$ is said to be an Intuitionistic Fuzzy Metric Space (for short, IFMS) if X is a vector space, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and μ, v are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x; y \in x$ and $s, t > 0$:

(1) $\mu(x, t) + v(x, t) \le 1$,
(2) $\mu(x, t) > 0$,
(3) $\mu(x, t) = 1$ if and only if $x = 0$,
(4) $\mu(\alpha x, t) = \mu\left(x, \dfrac{t}{|\alpha|}\right)$ for each $\alpha \ne 0$,
(5) $\mu(x, t) * \mu(y, s) \le \mu(x + y, t + s)$,
(6) $\mu(x, \cdot): (0, \infty) \to [0, 1]$ is continuous,
(7) $\lim_{t \to \infty} \mu(x, t) = 1$ and $\lim_{t \to 0} \mu(x, t) = 0$;
(8) $v(x, t) < 1$,
(9) $v(x, t) = 0$ and if and only if $x = 0$,
(10) $v(\alpha x, t) = v\left(x, \dfrac{t}{|\alpha|}\right)$ for each $\alpha \ne 0$,
(11) $v(x, t) \lozenge v(y, s) \ge v(x + y, t + s)$,
(12) $v(x, \cdot): (0, \infty) \to [0, 1]$ is continuous,
(13) $\lim_{t \to \infty} v(x, t) = 0$ and $\lim_{t \to 0} v(x, t) = 1$.

In this case (μ, v) is called intuitionistic fuzzy metric.

Definition 2.2

Let $(X, \mu, v, *, \lozenge)$ be an rough intuitionistic fuzzy metric space and β be a non negative real number. A triple sequence $x = (x_m nk)$ in X is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy metric (μ, v) if, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $m_0 n_0 k_0 \in \mathbb{N}$ such that $\mu(x_{mnk}\text{-}L, t) > 1\text{-}(\beta + \epsilon)$ and $v(x_{mnk}\text{-}L, t) < \beta + \epsilon$ for all $m \ge m_0$, $n \ge n_0$, $k \ge k_0$.

It is denoted by (μ, v)-lim $x = L$.

Definition 2.3

Let $(X, \mu, v, *, \lozenge)$ be an rough intuitionistic fuzzy metric space and β be a non negative real number. A triple sequence $x = (x_{mnk})$ in X is said to be a Cauchy sequence with respect to the rough intuitionistic fuzzy metric (μ, v) if, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $m_0 n_0 k_0 \in \mathbb{N}$ such that $\mu(x_{mnk}\text{-}x_{uvw}, t) > 1 - (\beta + \epsilon)$ and $v(x_{mnk}-x_{uvw}, t) < \beta + \epsilon$ for all $m, n, k, u, v, w \ge m_0 n_0 k_0$.

Definition 2.4

Let $(X, \mu, v, *, \lozenge)$ be an rough intuitionistic fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined as:

$$B(x, r, t) = \begin{Bmatrix} y \in X : \mu(x - y, t) \\ > 1 - r \text{ and } v(x - y, t) < r \end{Bmatrix}$$

Definition 2.5

If X be a non-empty set. Then a family $I \subset P(X)$ of subsets of X is called an ideal in X if and only if:

(i) $\emptyset \in I$
(ii) $A, B \in I$ implies $A \subset B \in I$
(iii) For each $A \in I$ and $B \subset A$ we have $B \in I$

where, $P(X)$ is the power set of X.

Definition 2.6

If X be a non-empty set. Then a non-empty family of sets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if:

(i) (i) $\emptyset \notin \mathcal{F}$
(ii) $A, B \in \mathcal{F}$ implies that $A \cap B \in \mathcal{F}$
(iii) For each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$

An ideal I is called non-trivial if $I \ne \emptyset$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if $I \supset \{\{x\}: x \in X\}$.

Remark 2.1

For each ideal I there is a filter $\mathcal{F}(I)$ which corresponding to I (filter associate with ideal I), that is:

$$\mathcal{F}(I)\{K \subseteq X : K^C \in I\}, \text{ where } K^C = X \setminus K. \qquad (2.1)$$

Definition 2.7

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that:

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty$$

and:

$$n_0 = 0, \ \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \ as \ \ell \to \infty.$$

$$k_0 = 0, \ \overline{h_j} = k_j - k_{j-1} \to \infty \ as \ j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j$, $h_{i,\ell,j} = h_i h_\ell h_j$ and $\theta_{i,\ell,j}$ be determined by:

$$I_{i,\ell,j} = \left\{ \begin{array}{l} (m,n,k): m_{i-1} < m < m_i \ and \ n_{\ell-1} \\ < n < n_\ell \ and \ k_{j-1} < k \le k_j \end{array} \right\},$$

$$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$$

Definition 2.8

Let $I \subset P(\mathbb{N})$ and let $(X, \mu, v, *, \Diamond)$ be an rough intuitionistic fuzzy metric space and β be a non negative real number. A triple sequence x = (x_{mnk}) in X is said to be Lacunary *I*-convergent to $L \in X$ with respect to the rough intuitionistic fuzzy metric (μ, v) if, for every $\epsilon \in (0, 1)$ and $t > 0$, the set:

$$\left\{ \begin{array}{l} r \in \mathbb{N}: \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x_{mnk} - L, t\right) \le \left(1 - (\beta + \epsilon)\right) \\ or \ \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(x_{mnk} - L, t\right) \ge \beta + \epsilon \end{array} \right\} \in I.$$

L is called the Lacunary *I*-limit of the triple sequence $x = (x_{mnk})$ and we write $I_\theta^{(\mu,v)}$-lim $x = L$.

Definition 2.9

Let $(X, \mu, v, *, \Diamond)$ be an rough intuitionistic fuzzy metric space and β be a non negative real number. A triple sequence $x = (x_{mnk})$ in X is said to be Lacunary *I*-Cauchy with respect to the rough intuitionistic fuzzy metric (μ, v) if, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $m \in \mathbb{N}$ satisfying:

$$\left\{ \begin{array}{l} r \in \mathbb{N}: \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x_{mnk} - x_{uvw}, t\right) \le \left(1 - (\beta + \epsilon)\right) \\ or \ \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(x_{mnk} - x_{uvw}, t\right) \ge \beta + \epsilon \end{array} \right\} \in I.$$

Zweier $I_\theta^{(\mu,v)}$-Convergent Sequence Spaces

The Zweier operator and I_θ-convergence of triple sequences in an rough intuitionistic fuzzy metric spaces and β be a non negative real number, we introduce the following new Zweier triple sequence spaces and examine some algebraic and topological properties on these spaces. Throughout the article, for the sake of convenience now we will denote by $Z^p(x_{mnk}) = x'$, $Z^p(y_{mnk}) = y'$, $Z^p(z_{mnk}) = z'$ where $x = (x_{mnk})$, $y = (y_{mnk})$, $z = (z_{mnk})$ are in ω:

$$Z_{(\mu,v)}^{I_\theta} = \left\{ x = (x_{mnk}) \in \omega : \left\{ r \in \mathbb{N}: \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x'_{mnk} - L, t\right) \le 1 - (\beta + \epsilon) \ or \ \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(x'_{mnk} - L, t\right) \ge \beta + \epsilon \right\} \in I \right\} \tag{3.1}$$

$$Z_{0(\mu,v)}^{I_\theta} = \left\{ x = (x_{mnk}) \in \omega : \left\{ r \in \mathbb{N}: \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x'_{mnk}, t\right) \le 1 - (\beta + \epsilon) \ or \ \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(x'_{mnk}, t\right) \ge \beta + \epsilon \right\} \in I \right\} \tag{3.2}$$

Also, we define an open ball with center x' and radius r with respect to t as follows:

$$B(x',r,t) = \left\{ y = (y_k) \in \omega : \left\{ r \in \mathbb{N}: \frac{1}{h_r} \sum_{k \in J_r} \mu(x' - y', t) \le 1 - (\beta + \epsilon) \ or \ \frac{1}{h_r} \sum_{k \in J_r} v(x' - y', t) \ge \beta + \epsilon \right\} \in I \right\} \tag{3.3}$$

Theorem 3.1

The space $Z_{(\mu,v)}^{I_\theta}$ and $Z_{0(\mu,v)}^{I_\theta}$ are linear space.

Proof

We prove the result for $Z_{0(\mu,v)}^{I_\theta}$. The proof for other space will follow Similarly. Let $x' = (x'_{mnk})$, $y' = (y'_{mnk}) \in Z_{(\mu,v)}^{I_\theta}$ and let α, γ be scalars. Then for a given $\epsilon > 0$, we have:

$$A_1 = \left\{ m,n,k \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x'_{mnk} - L_1, \frac{t}{2|\alpha|} \right) \le 1 - (\beta + \epsilon) \text{ or } \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(x'_{mnk} - L_1, \frac{t}{2|\alpha|} \right) \ge \beta + \epsilon \right\}$$

$$A_2 = \left\{ (m,n,k) \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(y'_{mnk} - L_2, \frac{t}{2|\gamma|} \right) \le 1 - \epsilon \text{ or } \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(y'_{mnk} - L_2, \frac{t}{2|\alpha|} \right) \ge \epsilon \right\} \in I,$$

$$A_1^c = \left\{ (m,n,k) \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(x'_{mnk} - L_1, \frac{t}{2|\gamma|} \right) > 1 - \left(\beta + \epsilon \text{ or } \frac{1}{h_{rst}} \sum_{mnk\in J_r} v\left(x'_{mnk} - L_1, \frac{t}{2|\alpha|} \right) < \beta + \epsilon \right) \in I \right.$$

$$A_2^c = \left\{ (m,n,k) \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(y'_{mnk} - L_2, \frac{t}{2|\gamma|} \right) > 1 - \epsilon \text{ or } \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} v\left(y'_{mnk} - L_2, \frac{t}{2|\alpha|} \right) < \beta + \epsilon \right\} \in F(I)$$

Define the set $A_3 = A_1 \cap A_2$, so that $A_3 \in I$. it follows that A_3^c is a non-empty set in $\mathcal{F}(I)$. We shall show that for each $(x'_{mnk}), (y'_{mnk}) \in Z^{I_\theta}(\mu, v)$.

Let $m \in A_3^I$. In this case:

$$\frac{1}{h_{rst,\ (abc)\in J_r}} \sum \mu\left(x'_{abc} - L_1 \frac{t}{2|\alpha|} \right) > 1 - (\beta + \epsilon)$$

$$or \frac{1}{h_{rst,\ (abc)\in J_r}} \sum v\left(x'_{abc} - L_1 \frac{t}{2|\alpha|} \right) < \beta + \epsilon$$

and:

$$\frac{1}{h_{rst,\ (abc)\in J_r}} \sum \mu\left(y'_{abc} - L_2 \frac{t}{2|\gamma|} \right) > 1 - (\beta + \epsilon)$$

$$or \frac{1}{h_{rst,\ (abc)\in J_r}} \sum v\left(y'_{abc} - L_2 \frac{t}{2|\gamma|} \right) < \beta + \epsilon$$

we have:

$$\frac{1}{h_{rst,\ (abc)\in J_r}} \sum \mu\left((\alpha x'_{abc} + \beta y'_{abc}) - (\alpha L_1 + \beta L_2), t \right)$$

$$\ge \frac{1}{h_{rst}} \sum_{(abc)\in J_r} \mu\left(\alpha x'_{abc} + \alpha L_1, \frac{t}{2} \right) * \frac{1}{h_{rst}} \sum_{abc\in J_r} \mu\left(\gamma y'_{abc} + \beta L_2, \frac{t}{2} \right)$$

$$= \frac{1}{h_{rst}} \sum_{(abc)\in J_r} \mu\left(x'_{abc} + L_1, \frac{t}{2|\alpha|} \right) * \frac{1}{h_{rst}} \sum_{(abc)\in J_r} \mu\left(y'_{abc} + L_2, \frac{t}{2|\gamma|} \right)$$

$$> (1 - (\beta + \epsilon)) * (1 - (\beta + \epsilon)) = (1 - (\beta + \epsilon)).$$

and:

$$\frac{1}{h_{rst,\ (abc)\in J_r}} \sum \mu\left((\alpha x'_{abc} + \beta y'_{abc}) - (\alpha L_1 + \gamma L_2), t \right)$$

$$\ge \frac{1}{h_{rst}} \sum_{(abc)\in J_r} v\left(\alpha x'_{abc} - \alpha L_1, \frac{t}{2} \right) \lozenge \frac{1}{h_{rst}} \sum_{(abc)\in J_r} v\left(\gamma y'_m - \gamma L_2, \frac{t}{2} \right)$$

$$= \frac{1}{h_{rst}} \sum_{(abc)\in J_r} v\left(x'_{abc} - L_1, \frac{t}{2|\alpha|} \right) \lozenge \frac{1}{h_{rst}} \sum_{(abc)\in J_r} v\left(y'_{abc} - L_2, \frac{t}{2|\gamma|} \right)$$

$$> \beta + \epsilon \lozenge \beta + \epsilon = \beta + \epsilon.$$

this is implies that:

$$A_3^c \subset \left\{ \begin{array}{l} (mnk) \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(mnk)\in J_r} \mu\left(\alpha x'_{mnk} + \gamma y'_{mnk} \right) - \left(\alpha L_1 + \gamma L_2 \right) \\ > 1 - (\beta + \epsilon) \\ or \frac{1}{h_{rst}} \sum_{((mnk)\in J_r} v\left(x'_{mnk} + \gamma y'_{mnk} \right) - (\alpha L_1 + \gamma L_2) < \beta + \epsilon \end{array} \right\}$$

Hence $Z^{I_\theta}_{(\mu,v)}$ is a linear space.

Theorem 3.2

Every open ball $B(x', t, \beta)$ is open in $Z^{I_\theta}_{(\mu,v)}$.

Proof

Let $B(x', t, \beta)$ be an open ball with center x' and radius β with respect to t. That is:

$$B(x', t, \beta)$$

$$= \left\{ \begin{array}{l} (m,n,k) \in \mathbb{N} : \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} \mu\left(x'_{mnk} - L, t \right) \le 1 - \beta \\ or \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} v\left(x'_{mnk} - L, t \right) \ge \beta \end{array} \right\} \in I.$$

Let $y' \in B^c(x', t, \beta)$. Then:

$$\frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} \mu(x' - y', t) > 1 - \beta \text{ and } \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} v(x' - y', t) < \beta$$

Since $\mu(x'-y', t) > 1-\beta$, there exists $t_0 \in (0, 1)$ such that:

$$\mu(x' - y', t_0) > 1 - \beta$$

$$and \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} v(x' - y', t_0) < \beta$$

Putting $\beta_0 = \frac{1}{h_{rst}} \sum_{(m,n,k)\in J_r} \mu(x' - y', t_0)$. We have $\beta_0 >$

$1-\beta$, there exists $s \in (0, 1)$ such that:

$$\beta_0 > 1 - s > 1 - \beta.$$

For $\beta_0 > 1-s$, we have $\beta_1, \beta_2 \in (0, 1)$ such that:

$$\beta_0 * \beta_1 > 1 - s \, and \left(1 - \beta_0\right) \Diamond \left(1 - \beta_2\right) \le s.$$

Putting $\beta_3 = \max\{\beta_1, \beta_2\}$, consider the ball $B^c(y', 1-\beta_3, t-t_0)$. We prove that:

$$B^c\left(y', 1-r_3, t-t_0\right) \subset B^c\left(x', r, t\right).$$

Let $z' \in B^c(y', 1-\beta_3, t-t_0)$:

$$\frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(y' - z', t-t_0\right) > \beta_3$$

$$and \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(y' - z', t-t_0\right) < \beta_3.$$

Therefore:

$$\frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(x' - z', t\right) \ge \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(x' - y', t_0\right)$$

$$* \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(y' - z', t-t_0\right)$$

$$\ge \left(\beta_0 * \beta_3\right) \ge \left(\beta_0 * \beta_1\right) \ge (1-s) > (1-\beta)$$

and:

$$\frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(x' - z', t\right)$$

$$\le \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(x' - y', t_0\right) \Diamond \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(y' - z', t-t_0\right)$$

$$\le \left(1-\beta_1\right) \Diamond \left(1-\beta_3\right) \le \left(1-\beta_0 > \left(1-\beta_2\right) < s < \beta\right.$$

Thus $z' \in B^c(x', t, \beta)$ and hence $B^c(y', 1-\beta_3, t-t_0 \subset B^c(x', t, \beta))$.

Remark 3.1.

$Z^{I_\theta}_{(\mu,v)}$ is RIFMS.

Define $\tau_{(\mu,v)} = \{A \subset Z^{I_\theta}_{(\mu,v)}$ for each $x \in A$, $\exists t > 0$ and $\beta \in (0, 1)$ s.t $B^c(x', t, \beta) \subset A\}$.

Then $\tau_{(\mu,v)}$ is topology on $Z^{I_\theta}_{(\mu,v)}$.

Theorem 3.3

The topology $\tau_{(\mu,v)}$ on $Z^{I_\theta}_{(\mu,v)}$ is first countable.

Proof

$\left\{B\left(x', \frac{1}{n^3}, \frac{1}{n^3}\right) : n = 1, 2, 3, \ldots\right\}$ is a local base at x' the topology $\tau_{(\mu,v)}$ is first countable.

Theorem 3.4.

$Z^{I_\theta}_{(\mu,v)}$ and $Z^{I_\theta}_{0(\mu,v)}$ are Hausdroff spaces.

Proof

We prove the result for $Z^{I_\theta}_{(\mu,v)}$. Similarly the result can be proved for $Z^{I_\theta}_{0(\mu,v)}$. Let $x', y' \in Z^{I_\theta}_{(\mu,v)}$ such that $x' \ne y'$. Then:

$$0 < \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(x' - y', t\right) < 1 \, and \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(x' - y', t\right) < 1.$$

Putting $\beta_1 = \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu(x'-y', t)$ and $\beta_2 = \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v(x'-y', t)$ and $r = \max\{\beta_1, 1-\beta_2\}$.

For each $\beta_0 \in (\beta, 1) \exists \beta_3$ and β_4 such that:

$$\beta_3 * \beta_4 \ge \beta_0 \, and \left(1-\beta_3\right) \Diamond \left(1-\beta_4\right) \le \left(1-\beta_0\right)$$

Putting $\beta_5 = \max\{\beta_3, 1-\beta_4\}$ and consider the open ball $B\left(x', 1-\beta_5 \frac{t}{2}\right)$ and $B\left(y', 1-\beta_5 \frac{t}{2}\right)$.

Clearly:

$$B\left(x', 1-\beta_5 \frac{t}{2} \cap B\left(y', 1-\beta_5 \frac{t}{2} = \phi\right)\right.$$

For if there exists $z' \in B\left(x', 1-\beta_5 \frac{t}{2} \cap B\left(y', 1-\beta_5 \frac{t}{2}\right)\right.$. Then and:

$$\beta_1 = \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(x' - y', t\right) \ge \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(x' - z', \frac{t}{2}\right)$$

$$* \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} \mu\left(z' - y', \frac{t}{2}\right)$$

$$\ge \beta_5 * \beta_5 \ge \beta_3 * \beta_3 \ge \beta_0 > \beta_1.$$

and:

$$\beta_2 = \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(x' - y', t\right)$$

$$\le \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(x' - z', \frac{t}{2}\right) \Diamond \frac{1}{h_{rst}} \sum_{(m,n,k) \in J_r} v\left(z' - y', \frac{t}{2}\right)$$

$$\le \left(1-\beta_5\right) \Diamond \left(1-\beta_5\right) \le \left(1-\beta_4\right) \Diamond \left(1-\beta_4\right)$$

$$\le \left(1-\beta_0\right) < \beta_0 > \beta,$$

which is contradiction. Hence $Z^{I_\theta}_{(\mu,v)}$ is Hausdroff space.

Conclusion

The triple sequence spaces of rough intuitionistic fuzzy zweier lacunary ideal convergence of sequence have been defined here and some algebraic and topological properties of these spaces are investigated. These definitions and results provide new tools to deal with the convergence problems of triple sequences in the fuzzy settings, occurring in many branches of science and engineering.

Authors Contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

Competing Interests

The authors declare that they have no competing interests.

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