Original Research Paper

Proportional Hazard Inverse Weibull Distribution and Associated Inference

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Abstract: In this study, we have considered the proportional hazard version of the inverse Weibull distribution. It has been shown that the density and hazard rate functions are unimodal and the mean residual life function is bathtub shaped. For the considered model, many functions of the parameters, such as the mean, variance, coefficient of variation and the critical points of the density, hazard rate and mean residual life functions cannot be given in explicit form. As a result, the variances of the maximum likelihood estimators of such functions cannot be given in explicit form to construct the usual asymptotic confidence interval. In this study, we use the percentile bootstrap estimation method to estimate such variances to construct the asymptotic confidence intervals. The asymptotic confidence intervals are compared with the bootstrap confidence intervals. Simulation studies are carried out to examine the performance of the maximum likelihood and bootstrap estimators. An example is provided to illustrate the procedure.

Keywords: Inverse Weibull Distribution, Moments, Probability Density Function, Hazard Rate Function, Mean Residual Life Function, Bootstrap Estimation

1. Introduction

The proportional hazard model has been widely used to analyze survival data. More specifically, let Y be a random variable with survival function $S_1(y)$ and hazard rate function $\lambda_1(y)$. Then a random variable X with survival function $S(x) = [S_1(x)]^\alpha$, $\alpha > 0$, has proportional hazard rate function $\lambda(x) = \alpha\lambda_1(x)$. The parameter α is called a fraility parameter, see for example Marshall and Olkin (2007).

The inverse Weibull (IW) distribution with survival function given by

$$S_1(y) = 1 - e^{-\lambda y^{-\beta}}, \ y > 0, \ \beta, \lambda > 0. \tag{1}$$

can be used to model a variety of failure characteristics such as infant mortality, useful life and wear out periods. For the structural properties of the IW distribution, we refer the reader Khan *et al.* (2008a; 2008b). The maximum likelihood and the least square estimations of the inverse Weibull distribution have been studied by Calabria and Pulcini (1990).

In various situations, it is more appropriate to employ the proportional hazard model than the original model. In this study, we are interested in the Proportional hazard Inverse Weibull (PHIW) distribution with survival function given by

$$S(x) = \left(1 - e^{-\lambda x^{-\beta}}\right)^\alpha, \ x > 0, \ \alpha, \beta, \lambda > 0. \tag{2}$$

Note that when $\alpha = 1$, the PHIW distribution reduces to the IW distribution. Also, when $\beta = 1$, the PHIW distribution reduces to the Generalized Inverted Exponential (GIE) distribution studied by Abouammoh and Alshingiti (2009).

It may be noted that analogues to the proportional hazard model, the proportional reversed hazard rate model has also been studied in the literature. In this case, the power of the distribution function is considered instead of the power of the survival function. For more information on the proportional reversed hazard rate model, we refer the reader to Gupta and Gupta (2007).

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The main purpose of this paper is to study the bootstrap estimation of the parameters of the PHIW distribution and the associated functions including the mean, variance, coefficient of variation, the probability density, the hazard rate and mean residual life functions. The paper is organized as follows: In section 2, we consider the shape of the density, hazard rate and mean residual life functions. The moments (including the negative moments) and the associated measures are given in section 3. In section 4, we study the maximum likelihood and bootstrap estimation of functions of the parameters. Simulation studies are carried out in section 5 to investigate the performance of the estimators of various functions of the parameters. An application is provided in section 6. Finally, some conclusion and comments are given.

2. Probability Density, Hazard Rate and Mean Residual Life Functions

2.1 Probability Density Function

The Probability Density Function (PDF) of the PHIW distribution, with survival function (2), is given by:

$$f(x) = \alpha\beta\lambda\left(1 - e^{-\lambda x^{-\beta}}\right)^{\alpha-1} x^{-\beta-1} e^{-\lambda x^{-\beta}}, \ x > 0, \alpha, \beta, \lambda > 0. \quad (3)$$

The following theorem investigates the shape of the PDF of the PHIW distribution.

Theorem 1

The PDF $f(x)$ is unimodal for all $\alpha, \beta, \lambda > 0$, with $f(0) = f(\infty) = 0$.

Proof.

It is clear that $f(0) = f(\infty) = 0$. The first derivative of $f(x)$ can be written in the form

$$f'(x) = \frac{f(x)}{x\left(1 - e^{-\lambda x^{-\beta}}\right)} \phi(x)$$

where,

$$\phi(x) = \left[\beta + 1 - \alpha\beta y(x)\right] e^{-y(x)}$$
$$+ \beta y(x) - (\beta + 1), \ y(x) = \lambda x^{-\beta}.$$

Note that

$$\phi'(x) = -y'(x)w(x)$$

where

$$w(x) = \left[\alpha\beta + \beta + 1 - \alpha\beta(x)\right] e^{-y(x)} - \beta.$$

In the following, we show that $w(x)$ changes sign from negative to positive. Since

$$w'(x) = -y'(x)\left[2\alpha\beta + \beta + 1 - \alpha\beta y(x)\right] e^{-y(x)}$$

it follows that $x_0 = [\alpha\beta\lambda/(2\alpha\beta + \beta + 1)]^{1/\beta}$ is a unique critical point where $w(x)$ has an absolute minimum, since

$$w''(x_0) = \alpha\beta\left[y'(x_0)\right]^2 e^{-y(x_0)} > 0.$$

Since $w(0) = -\beta$ and $w(\infty) = \alpha\beta + 1$, it follows that $w(x)$ changes sign from negative to positive. This implies that $\phi'(x)$ also changes sign from negative to positive, since $-y'(x) > 0$. Finally, since $\phi(0) = \infty$ and $\phi(\infty) = 0$, $\phi(x)$ decreases from infinity to a negative value then increases to zero, i.e., $\phi(x)$ changes sign from positive to negative. That is, $f'(x)$ changes sign from positive to negative proving the unimodality of $f(x)$.

Remarks

- The mode, x_f, is the solution of the non-linear equation $f'(x_f) = 0$ or equivalently

$$\phi(x_f) = \left[\beta + 1 - \alpha\beta y(x_f)\right] e^{-y(x_f)}$$
$$+ \beta y(x_f) - (\beta + 1) = 0, y(x_f) = \lambda x_f^{-\beta}. \quad (4)$$

- For $\alpha = 1, \beta, \lambda > 0$, i.e., the IW distribution, the mode x_f has an explicit closed form: $x_f = [\beta\lambda/(\beta + 1)]^{1/\beta}$, see Jiang *et al.* (2001).
- For $\alpha, \beta, \lambda > 0$, since $\phi(0) = \infty$ and

$$\phi(u_1) = \phi\left[\left(\lambda / \log(\alpha + 1)\right)^{1/\beta}\right] =$$
$$-\frac{1}{\alpha}\left\{\alpha + \beta\left[\alpha - \log(\alpha + 1)\right]\right\} < 0$$

since $\log(\alpha + 1) < \alpha$, for all $\alpha > 0$, the search for the mode x_f can be limited to the interval $(0, u_1)$. For example, see section 6, $\alpha = 16.391, \beta = 3.098, \lambda = 44.632, u_1 = 2.429$ and $x_f = 2.321 \in (0, u_1)$.

Figure 1 shows the PDF of the PHIW distribution for selected values of the parameters.

2.2 Hazard Rate Function

The Hazard Rate Function (HRF) of the PHIW distribution, with survival function (2) and PDF (3), is given by

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta\lambda x^{-\beta-1}e^{-\lambda x^{-\beta}}}{1 - e^{-\lambda x^{-\beta}}}, x > 0, \alpha, \beta, \lambda > 0. \qquad (5)$$

The following theorem investigates the shape of the HRF of the PHIW distribution.

Theorem 2

The HRF $h(x)$ is unimodal (upside-down bathtub shape) for all $\alpha, \beta, \lambda > 0$, with $h(0) = h(\infty) = 0$.

Proof.

It is clear that $h(0) = h(\infty) = 0$. Since $h(x) = \alpha\, h_1(x)$ where

$$h_i(x) = \frac{\beta\lambda x^{-\beta-1}e^{-\lambda x^{-\beta}}}{1 - e^{-\lambda x^{-\beta}}}, x > 0, \beta, \lambda > 0,$$

is the HRF of the IW distribution with PDF

$$f_1(x) = \beta\lambda x^{-\beta-1}e^{-\lambda x^{-\beta}}, x > 0, \beta, \lambda > 0,$$

and survival function $S_1(x)$. Clearly, the shape of $h(x)$ is the same of $h_1(x)$. Now, we use Glaser (1980) eta function of $f_1(x)$:

$$\eta(x) = -\left(\log f_1(x)\right)' = (\beta+1)x^{-1} - \beta\lambda x^{-\beta-1},$$

to determine the shape of $h_1(x)$. Since

$$\eta'(x) = (\beta+1)x^{-\beta-2}\left(\beta\lambda - x^{\beta}\right),$$

it follows that $u_2 = (\beta\lambda)^{1/\beta}$ is the unique critical point which maximizes $\eta(x)$, since

$$\eta''(u_2) = -(\beta+1)\beta^2\lambda u_2^{-\beta-3} < 0.$$

That is, $\eta(x)$ is unimodal with $\eta(0) = -\infty$ and $\eta(\infty) = 0$. Finally, since $f_1(0) = 0$, it follows that $h_1(x)$ and hence $h(x)$, is also unimodal.

Remarks

The critical point, x_h, of $h(x)$ is the unique solution of $h'(x_h) = 0$, which is equivalent to solving the non-linear equation

$$\psi(x_h) = (\beta+1)e^{-y(x_h)} + \beta y(x_h)$$
$$-(\beta+1) = 0, y(x_h) = \lambda x_h^{-\beta} \qquad (6)$$
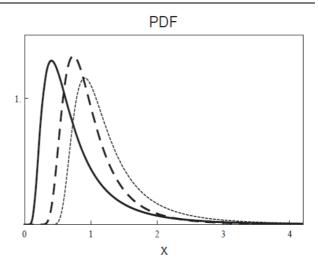
which is independent of α.

Fig. 1. PDF of the PHIW distribution for (α, β, λ): (3,1,1)(solid), (2,2,1)(dashed) and (1,3,1) (dotted)
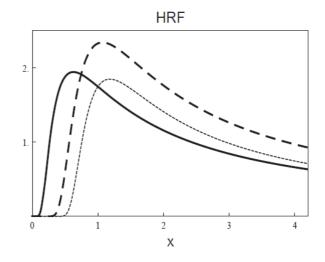

Fig. 2. HRF of the PHIW distribution for (α, β, λ): (3,1,1)(solid), (2,2,1) (dashed) and (1,3,1) (dotted)

It is clear that $\psi(0) = \infty$ and $\psi(u_2) = \psi[(\beta\lambda)^{1/\beta}] = -\beta$ $[1 - (1 + 1/\beta)e^{-1/\beta}] < 0$, since $(1 + z)e^{-z} < 1$, for all $z > 0$. This will be useful in finding the unique root x_h of $\psi(x_h)$ by limiting the search on the interval $(0, u_2)$. For example, see section 6, $\alpha = 16.391$, $\beta = 3.098$, $\lambda = 44.632$, $u_2 = 4.909$ and $x_h = 4.045 \in (0, u_2)$.

Figure 2 shows the HRF of the PHIW distribution for selected values of the parameters.

2.3 Mean Residual Life Function

Using the substitution $y = \lambda x^{-\beta}$, the mean of the PHIW distribution is given by

$$\mu = \int_0^\infty S(x)dx = \frac{\lambda^{1/\beta}}{\beta}\int_0^\infty y^{-1/\beta-1}\left(1 - e^{-y}\right)^{\alpha} dy.$$

For $0 < \alpha\beta \leq 1$, using the inequality $1 - e^{-y} \geq y\, e^{-y}$, for all $y \geq 0$, we have

$$\mu \geq \frac{\lambda^{1/\beta}}{\beta} \int_0^\infty y^{(\alpha\beta-1)\beta-1} e^{-\alpha y} dy = \infty.$$

For $\alpha\beta > 1$, using the inequalities $1 - e^{-y} \leq y$ and $1 - e^{-y} \leq 1$, for all $y \geq 0$, respectively, we have

$$\mu \geq \frac{\lambda^{1/\beta}}{\beta} \left\{ \int_0^1 y^{-1/\beta-1} \left(1-e^{-y}\right)^\alpha dy + \int_1^\infty y^{-1/\beta-1} \left(1-e^{-y}\right)^\alpha dy \right\}$$

$$\leq \frac{\lambda^{1/\beta}}{\beta} \left\{ \int_0^1 y^{(\alpha\beta-1)/\beta-1} dy + \int_1^\infty y^{-1/\beta-1} dy \right\}$$

$$= \frac{\alpha\beta\lambda^{1/\beta}}{\alpha\beta-1} < \infty.$$

For all $x > 0$, the Mean Residual Life Function (MRLF) of the PHIW distribution, with survival function (2) and PDF (3), is given by

$$\mu(x) = E(X - x \mid X > x)$$

$$= \frac{1}{S(x)} \int_x^\infty y f(y) dy - x$$

$$= \frac{\alpha\beta\lambda}{\left(1-e^{-\lambda x^{-\beta}}\right)^\alpha} \int_x^\infty \left(1-e^{-\lambda y^{-\beta}}\right)^{\alpha-1} y^{-\beta} e^{-\lambda y^{-\beta}} dy - x.$$

Note that $\mu(0) = \mu < \infty$, provided $\alpha\beta > 1$ *and* $\mu(\infty) = 1/h(\infty) = \infty$.

For $\alpha = 1, 2, 3, \ldots$ and $\beta > 1$, using the substitution $z = \lambda y^{-\beta}$, we have

$$\mu(x) = \frac{\alpha\lambda^{1/\beta}}{\left(1-e^{-\lambda x^{-\beta}}\right)^\alpha} \int_0^{\lambda x^{-\beta}} \left(1-e^{-z}\right)^{\alpha-1} z^{-1/\beta} e^{-z} dz - x$$

$$= \frac{\alpha\lambda^{1/\beta}}{\left(1-e^{-\lambda x^{-\beta}}\right)^\alpha} \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i \int_0^{\lambda x^{-\beta}} z^{-1/\beta} e^{-(i+1)z} dz - x$$

$$= \frac{\alpha\lambda^{1/\beta}}{\left(1-e^{-\lambda x^{-\beta}}\right)^\alpha} \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} \frac{(-1)^i}{(i+1)^{1-1/\beta}} \gamma\left(1-1/\beta, (i+1)\lambda x^{-\beta}\right) - x.$$

where, $\gamma(a,t) = \int_0^t u^{a-1} e^{-u} du, a > 0$, is the lower incomplete gamma function for which $\gamma(a, 0) = 0$ and $\gamma(a,\infty) = \Gamma(a)$ is the gamma function.

For the special case, $\alpha = 1$, i.e., IW distribution, we have

$$\mu(x) = \frac{\lambda^{1/\beta} \gamma\left(1-1/\beta, \lambda x^{-\beta}\right)}{1-e-\lambda x^{-\beta}} - x, \ \beta > 1.$$

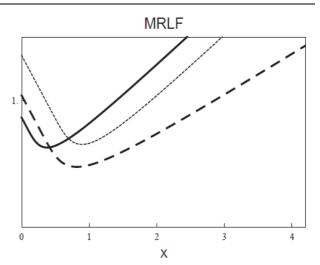Before we determine the shape of $\mu(x)$, we need the following lemma.

Fig. 3. MRLF of the PHIW distribution for (α, β, λ): (3,1,1) (solid), (2,2,1)(dashed) and (1,3,1) (dotted)

Lemma 1

Gupta and Akman (1995) If $h(x)$ is upside-down bathtub shape with a unique critical point x_h and $f(0)\mu < 1$, then $\mu(x)$ is bathtub shape with a unique critical point x_m. The critical point x_m is the solution of the non-linear equation

$$h(x_m)\mu(x_m) - 1 = 0. \tag{7}$$

Moreover, $x_m < x_h$.

Using the condition $\alpha\beta > 1$ and Lemma 1, we are now able to determine the shape of the MRLF of the PHIW distribution as follows.

Theorem 3

The MRLF $\mu(x)$ is upside-down bathtub shape with unique critical point x_m for all $\alpha, \beta > 0$ such that $\alpha\beta > 1$ and $\lambda > 0$.

Proof.

Since $\alpha\beta > 1$ implies that $\mu < \infty$ *and* $f(0) = 0$, we have $f(0)\mu = 0 < 1$. Therefore, using Lemma 1, the MRLF is bathtub shaped.

Figure 3 shows the MRLF of the PHIW distribution for selected values of the parameters.

3. Moments and Associated Measures

The kth raw moment of the PHIW distribution is given by

$$\mu'_k = \int_0^\infty k\, x^{k-1} S(x) dx = \frac{k\lambda^{k/\beta}}{\beta} \int_0^\infty y^{-k/\beta-1} \left(1-e^{-y}\right)^\alpha dy.$$

By exactly the same steps as in subsection 2.3, μ'_k exists only when $\alpha\beta > k$. For positive real values α and β, the moments cannot be expressed in closed form expressions. However, for $\alpha = 1, 2, 3, \ldots$, we have

$$\mu'_k = \alpha\lambda^{k/\beta}\Gamma\left(1 - k/\beta\right)\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}\frac{(-1)^i}{(i+1)^{1-k/\beta}}, \beta > k.$$

For the special case, $\alpha = 1$, i.e., IW distribution, we have:

$$\mu'_k = \lambda^{k/\beta}\Gamma\left(1 - k/\beta\right), \beta > k.$$

In this case, the mean, variance and coefficient of variation, respectively, are

$$\mu = \lambda^{1/\beta}\Gamma\left(1 - 1/\beta\right), \beta > 1,$$
$$\sigma^2 = \lambda^{2/\beta}\left[\Gamma\left(1 - 2/\beta\right) - \Gamma^2\left(1 - 1/\beta\right)\right], \beta > 2,$$
$$cv = \frac{\sqrt{\Gamma\left(1 - 2/\beta\right) - \Gamma^2\left(1 - 1/\beta\right)}}{\Gamma\left(1 - 1/\beta\right)}, \beta > 2.$$

Note that if $X \sim \text{PHIW}(\alpha, \beta, \gamma)$, then $Z = 1/X$ has the exponentiated Weibull distribution (Mudholkar and Srivastava 1993) with cumulative distribution function

$$F_Z\left(z\right) = \left(1 - e^{-\lambda z^\beta}\right)^\alpha, z > 0, \alpha, \beta, \gamma > 0.$$

Now the negative moments of the PHIW distribution are given by

$$E\left(X^{-k}\right) = E\left(Z^k\right)$$
$$= \alpha\lambda^{-k/\beta}\Gamma\left(k/\beta + 1\right)\sum_{i=0}^{\infty}\frac{(1-\alpha)}{i!(i+1)^{k/\beta+1}}$$

where, for any real number c:

$$\left(c\right)_i = \begin{cases} 1, & \text{if } i = 0 \\ c(c+1)\ldots(c+i-1), & \text{if } i \geq 1 \end{cases}$$

is the Pochhammer symbol, for more details see Choudhury (2005).

For $\alpha = 1, 2, \ldots$, we have:

$$E\left(X^{-k}\right) = \alpha\lambda^{-k/\beta}\Gamma\left(k/\beta + 1\right)\sum_{i=0}^{\alpha-1}\frac{(1-\alpha)_i}{i!(i+1)^{k/\beta+1}}.$$

In particular, if $\alpha = 1$, i.e., the IW distribution, we have

$$E\left(X^{-k}\right) = \lambda^{-k/\beta}\Gamma\left(k/\beta + 1\right).$$

4. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the PHIW distribution with PDF (3). The log-likelihood function is given by

$$\ell_n\left(\theta\right) = \sum_{i=1}^{n}\log f\left(x_i\right),$$
$$= n\log\left(\alpha\beta\lambda\right) - \sum_{i=1}^{n}\left\{\begin{array}{l}(\beta+1)\log\left(x_i\right) \\ +\lambda x_i^{-\beta} - (\alpha-1)\log\left(1 - e^{-\lambda x_i^{-\beta}}\right)\end{array}\right\}.$$

The maximum likelihood estimates (MLEs) $\left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ of the parameters (α, β, λ) are the solution of the system of non-linear equations

$$\frac{\partial\ell_n\left(\theta\right)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1 - e^{-\lambda x_i^{-\beta}}\right) = 0, \quad (8)$$

$$\frac{\partial\ell_n\left(\theta\right)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left\{1 - \lambda x_i^{-\beta} + \frac{(\alpha-1)\lambda x_i^{-\beta}}{e^{\lambda x_i^{-\beta}} - 1}\right\}\log\left(x_i\right) = 0, \quad (9)$$

$$\frac{\partial\ell_n\left(\theta\right)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\left\{-1\frac{(\alpha-1)}{e^{\lambda x_i^{-\beta}} - 1}\right\}\log\left(x_i^{-\beta}\right) = 0. \quad (10)$$

Note that (8) implies that

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\log\left(1 - e^{-\hat{\lambda} x_i^{-\hat{\beta}}}\right)} = \xi\left(\hat{\beta}, \hat{\lambda}\right). \quad (11)$$

Hence, the 3-dimensional maximization in (α, β, λ) is reduced to 2-dimensional maximization in (β, λ):

$$\frac{n}{\beta} - \sum_{i=1}^{n}\left\{1 - \lambda x_i^{-\beta} + \frac{\left[\xi(\beta,\lambda)-1\right]\lambda x_i^{-\beta}}{e^{\lambda x_i^{-\beta}} - 1}\right\}\log\left(x_i\right) = 0, \quad (12)$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n}\left\{1 - \frac{\xi(\beta,\lambda)-1}{e^{\lambda x_i^{-\beta}} - 1}\right\}x_i^{-\beta} = 0. \quad (13)$$

The solutions of the last two nonlinear equations provide the MLEs $\hat{\beta}$ and $\hat{\lambda}$ of the parameters β and λ, from which the MLE $\hat{\alpha} = \xi\left(\hat{\beta}, \hat{\lambda}\right)$ of the parameter α is obtained.

The observed information matrix is

$$I_n\left(\theta\right) = \left[-\frac{\partial^2\ell_n\left(\theta\right)}{\partial\theta_i\partial\theta_j}\right] = \begin{bmatrix} I_n^{11}\left(\theta\right) & I_n^{12}\left(\theta\right) & I_n^{13}\left(\theta\right) \\ I_n^{12}\left(\theta\right) & I_n^{22}\left(\theta\right) & I_n^{23}\left(\theta\right) \\ I_n^{12}\left(\theta\right) & I_n^{23}\left(\theta\right) & I_n^{33}\left(\theta\right) \end{bmatrix}$$

where

$$I_n^{11}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \alpha^2} = \frac{n}{\alpha^2},$$

$$I_n^{22}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \beta^2} = \frac{n}{\beta^2}$$

$$+ \lambda \sum_{i=1}^{n} \left\{ 1 + (\alpha-1) \frac{1 - (1 - \lambda x_i^{-\beta}) e^{\lambda x_i^{-\beta}}}{(e^{\lambda x_i^{-\beta}} - 1)^2} \right\} x_i^{-\beta} \left[\log(x_i) \right]^2,$$

$$I_n^{33}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \lambda^2} = \frac{n}{\lambda^2} - (\alpha-1) \sum_{i=1}^{n} \frac{x_i^{-2\beta} e^{\lambda x_i^{-\beta}}}{(e^{\lambda x_i^{-\beta}} - 1)^2},$$

$$I_n^{12}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \alpha \partial \beta} = \lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} \log(x_i)}{e^{\lambda x_i^{-\beta}} - 1},$$

$$I_n^{13}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{n} \frac{x_i^{-\beta}}{e^{\lambda x_i^{-\beta}} - 1},$$

$$I_n^{23}(\theta) = -\frac{\partial^2 \ell_n(\theta)}{\partial \beta \partial \lambda} =$$

$$-\sum_{i=1}^{n} \left\{ 1 + (\alpha-1) \frac{1 - (1 - \lambda x_i^{-\beta}) e^{\lambda x_i^{-\beta}}}{(e\lambda x_i^{-\beta} - 1)^2} \right\} x_i^{-\beta} \log(x_i).$$

Under fairly mild regularity conditions, see Lehmann and Casella (1998), pp. 461-463, we have the following result

$$\hat{\theta} \underset{d}{\sim} N_3 \left(\theta, I_n^{-1}(\hat{\theta}) \right),$$

where, $\underset{d}{\sim}$ means approximately distributed and $N_3(., .)$ denotes the trivariate normal distribution.

For a real-valued function of the parameters $\tau(\theta)$, we have

$$\tau(\hat{\theta}) \underset{d}{\sim} N \left(\tau(\theta), \text{var}\left[\tau(\hat{\theta}) \right] \right).$$

The asymptotic 100(1-α)% confidence interval for τ (θ) is given by

$$\tau(\hat{\theta}) \pm z_{\alpha/2} \sqrt{\widehat{\text{var}\left[\tau(\hat{\theta}) \right]}}.$$

where, $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal distribution.

When the function $\tau(\theta)$ has no explicit expression in the parameters, as the case in this study, then $\text{var}\left[\tau(\hat{\theta}) \right]$ has also no explicit expression and is usually estimated using simulation-based methods such as percentile bootstrap, see for example Davison and Hinkley (1997), where B random samples are selected with replacement from the given data set. For the jth bootstrap sample, the

$MLE \hat{\theta}_j^*$ of θ and the $MLE \hat{\tau}_j^* = \tau(\hat{\theta}_j^*)$ of τ (θ) are calculated. The mean and variance of all bootstrap estimates $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$, respectively, are given by

$$\bar{\tau}^* = \frac{1}{B} \sum_{j=1}^{B} \hat{\tau}_j^*,$$

$$s^{*2} = \frac{1}{B-1} \sum_{j=1}^{B} \left(\hat{\tau}_j^* - \bar{\tau}^* \right)^2.$$

Now, we can use $\bar{\tau}^*$ and s^{*2} as estimates of τ (θ) and $\text{var}\left[\tau(\hat{\theta}) \right]$, respectively. Moreover, in this case, the following confidence intervals can be constructed.

The asymptotic 100(1-α)% confidence interval for τ (θ) is given by

$$\tau(\hat{\theta}) \pm z_{\alpha/2} s^*.$$

The 100(1-α)% percentile bootstrap confidence interval for τ (θ) is given by

$$\left(\hat{\tau}^{*(\alpha/2)}, \hat{\tau}^{*(1-\alpha/2)} \right)$$

where, $\hat{\tau}^{*(p)}$ is the pth percentile of the bootstrap estimates $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$.

5. Simulation Studies

Simulation of an observation x from the PHIW distribution is obtained by solving the equation $S(x) = 1-u$ where u is an observation from the uniform distribution over (0,1). It follows that

$$x = \left\{ \frac{-\lambda}{\log\left[1 - (1-u)^{1/\alpha} \right]} \right\}^{1/\beta}. \tag{14}$$

Design of the Simulation Studies

- Sample sizes: $n = 50, 100$
- Population parameters: (α, β, λ) = (10, 5, 1), (5, 10, 1). Table 1 shows the true values of the parameters used in the simulations
- Number of bootstrap samples: $B = 500$
- Number of simulation runs $N = 500$

Computations

- All computations in the simulation studies were done using the R package Version 3.1.2, see RDCT (2014)
- Maximum likelihood estimates of the parameters were obtained using the R general purpose

optimization function *optim* which implements the Nelder-Mead technique

- The bootstrap samples were obtained using the R package boot by Canty and Ripley (2015)

Simulation Criteria

Let $\tilde{\tau} \equiv \tau(\tilde{\theta})$ be the MLE or bootstrap estimate of $\tau \equiv \tau(\theta)$ and $\tilde{\tau}_{[i]} \equiv \tau(\tilde{\theta}_{[i]})$ be the estimate of τ in the ith simulation run, $i = 1, 2, \ldots, N$.

The following criteria are used to assess the performance of the MLEs/bootstrap estimates of $\tau \equiv \tau(\theta)$ and associated confidence intervals.

- Bias:

$$Bias(\tilde{\tau}) = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{\tau}_{[i]} - \tau\right)$$

- Mean-square error (MSE):

$$MSE(\tilde{\tau}) = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{\tau}_{[i]} - \tau\right)^2$$

- Coverage probability (CP): Percentage of the true value of the parameter which falls in a given confidence interval
- Confidence interval width: The range of the confidence interval

Analysis of the Simulation Studies

- For the mean, variance and coefficient of variation (Table 2), both confidence intervals are performing almost the same in terms of the coverage probability as well as their width
- In the case of the estimation of critical points of the PDF, HRF and MRLF (Table 3), the coverage probability of both confidence intervals is the same while the bootstrap intervals are outperforming the asymptotic intervals in terms of their width

Table 1. True values of the functions of the parameters used in the simulations

(α, β, λ)	μ	σ^2	cv	x_f	x_h	x_m
(5,10,1)	0.9657	0.0025	0.0539	0.9243	1.1784	1.1555
(10,5,1)	0.8232	0.0047	0.0832	0.8130	1.2158	1.1888

Table 2. Simulation results for the estimation of mean, variance and coefficient of variation

(α, β, λ)	n	Method	Bias	MSE	CP (%)	C.I. width
Mean						
(5, 10, 1)	50	Asymptotic	0.0001	0.0000	95.40	0.0275
		Bootstrap	0.0001	0.0000	94.80	0.0277
	100	Asymptotic	-0.0001	0.0000	95.20	0.0196
		Bootstrap	-0.0001	0.0000	95.00	0.0198
(10, 5, 1)	50	Asymptotic	0.0006	0.0001	93.40	0.0373
		Bootstrap	0.0006	0.0001	92.80	0.0376
	100	Asymptotic	0.0002	0.0000	95.80	0.0268
		Bootstrap	0.0003	0.0000	95.80	0.0270
Variance						
(5, 10, 1)	50	Asymptotic	-0.0001	0.0000	88.40	0.0022
		Bootstrap	-0.0001	0.0000	89.20	0.0022
	100	Asymptotic	-0.0001	0.0000	92.20	0.0016
		Bootstrap	-0.0001	0.0000	92.40	0.0016
(10, 5, 1)	50	Asymptotic	-0.0001	0.0000	89.20	0.0040
		Bootstrap	-0.0001	0.0000	89.40	0.0040
	100	Asymptotic	0.0000	0.0000	94.00	0.0028
		Bootstrap	0.0000	0.0000	94.00	0.0028
Coefficient of variation						
(5, 10, 1)	50	Asymptotic	-0.0007	0.0000	91.00	0.0231
		Bootstrap	-0.0011	0.0000	89.40	0.0233
	100	Asymptotic	-0.0004	0.0000	92.80	0.0165
		Bootstrap	-0.0006	0.0000	92.80	0.0166
(10, 5, 1)	50	Asymptotic	-0.0013	0.0001	93.40	0.0340
		Bootstrap	-0.0020	0.0000	92.40	0.0351
	100	Asymptotic	-0.0002	0.0000	94.80	0.0242
		Bootstrap	-0.0006	0.0000	94.00	0.0244

Table 3. Simulation results for the estimation of critical points of PDF, HRF and MRLF

(α, β, λ)	n	Method	Bias	MSE	CP (%)	C.I. width
Critical point of PDF						
(5, 10, 1)	50	Asymptotic	0.0576	0.1949	96.20	2.0431
		Bootstrap	0.1403	0.0517	97.20	1.6196
	100	Asymptotic	0.0167	0.0662	96.20	1.0865
		Bootstrap	0.0545	0.0174	95.40	0.6484
(10, 5, 1)	50	Asymptotic	0.0097	0.0351	93.00	0.4479
		Bootstrap	0.0137	0.0008	94.80	0.1729
	100	Asymptotic	0.0005	0.0001	95.20	0.0971
		Bootstrap	0.0020	0.0001	96.00	0.0536
Critical point of HRF						
(5, 10, 1)	50	Asymptotic	0.1153	0.2108	95.60	2.2538
		Bootstrap	0.2312	0.0731	96.00	2.2134
	100	Asymptotic	0.0414	0.0724	99.00	1.5893
		Bootstrap	0.1183	0.0264	97.20	1.2926
(10, 5, 1)	50	Asymptotic	0.1325	0.1887	95.80	1.8136
		Bootstrap	0.2718	0.1447	97.80	1.6402
	100	Asymptotic	0.0562	0.0514	98.60	1.0586
		Bootstrap	0.1104	0.0616	95.40	0.9683
Critical point of MRLF						
(5, 10, 1)	50	Asymptotic	0.1518	0.3011	96.20	2.8704
		Bootstrap	0.3851	0.2172	96.40	2.7247
	100	Asymptotic	0.0642	0.1435	98.80	1.9561
		Bootstrap	0.1859	0.0909	97.00	1.6775
(10, 5, 1)	50	Asymptotic	0.0884	0.1622	97.20	2.0358
		Bootstrap	0.2055	0.0920	98.00	1.9357
	100	Asymptotic	0.0472	0.0440	99.20	1.0277
		Bootstrap	0.0803	0.0365	96.40	0.8916

Thus, overall, the simulation studies indicate that both kind of confidence intervals are performing similar in terms of the coverage probabilities as well as their width.

Figure 4 shows approximately normal distribution for the sampling distribution of the MLEs of the mean, variance and coefficient of variation, indicating that the corresponding constructed asymptotic confidence intervals are appropriate. On the other hand, Figure 5 shows skewed sampling distribution of the MLEs of the critical point of the PDF, HRF and MRLF, indicating that the constructed asymptotic confidence intervals are inappropriate.

6. Illustrative Example

In this section, we present an illustrative example based on a real-life data set representing 98 measurements of the annual maxima of river flows ($m^3 s^{-1}$) from 1900 to 1997 at BSWN station in Shunde area, Pearl River basin, China (Shao *et al.*, 2004):

1.678, 2.285, 2.371, 2.255, 2.275, 2.14, 2.464,
2.052, 2.542, 2.355, 1.957, 2.32, 2.487, 2.358,
2.567, 3.123, 2.052, 2.464, 2.566, 2.438,
2.438, 2.285, 2.409, 2.499, 2.595, 2.076,
2.404, 2.261, 2.267, 2.189, 2.362, 2.675,

2.435, 2.438, 2.261, 2.552, 2.438, 2.567, 2.07,
2.288, 2.46, 2.303, 2.512, 2.33, 2.729, 2.736,
2.381, 2.762, 2.249, 2.862, 2.376, 2.174, 1.89,
1.93, 2.16, 1.92, 2.05, 2.03, 1.91, 2.62, 1.95,
2.34, 2.67, 2.14, 2.24, 2.4, 2.63, 2.25, 2.8,
2.26, 2.58, 2.22, 2.09, 2.46, 2.96, 2.47, 2.7,
2.32, 2.76, 2.3, 1.95, 2.33, 2.13, 2.54, 2.01,
2.06, 2.3, 1.98, 2.34, 2.43, 1.91, 2.44, 2.29,
2.53, 3.21, 2.27, 2.46, 2.77.

A summary of the basic descriptive sample statistics of this data set is given in Table 4.

Table 5 shows the MLEs of the parameters, the estimated log-likelihood, Kolomogrov-Smirnov and Anderson-Darling goodness-of-fit tests of the fitted PHIW model. Both tests indicate that the fitted PHIW model is suitable for the given data set. This conclusion is also supported by the diagnostic plots presented in Fig. 6.

Figure 7 shows the location of the percentile bootstrap estimates of the critical points of the PDF, HRF and MRLF of the fitted PHIW model. The histograms of the percentile bootstrap estimates of the mean, variance, coefficient of variation and the critical points of the PDF, HRF and MRLF are presented in Fig. 8 which indicates that the histograms of the critical points are heavily skewed to the right.
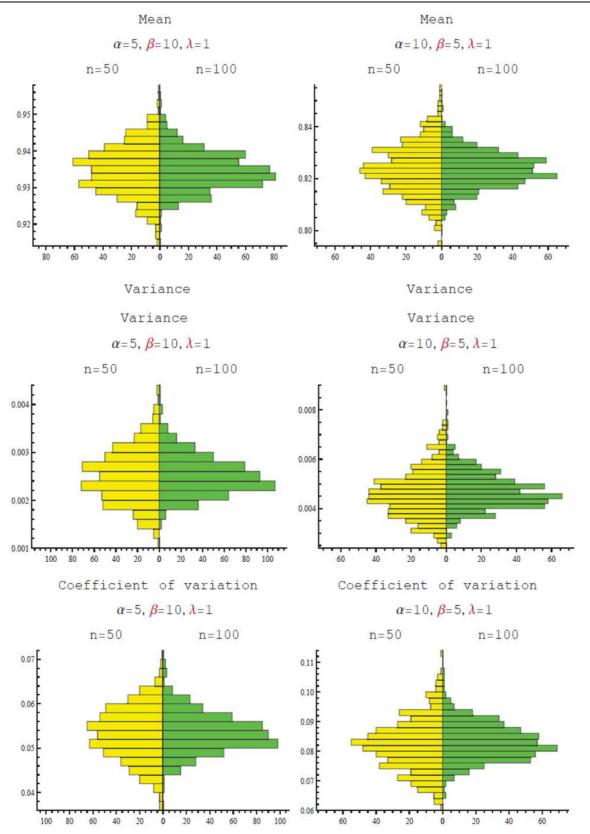
Fig. 4. Histograms of the sampling distributions of the MLEs of the mean, variance and coefficient of variation
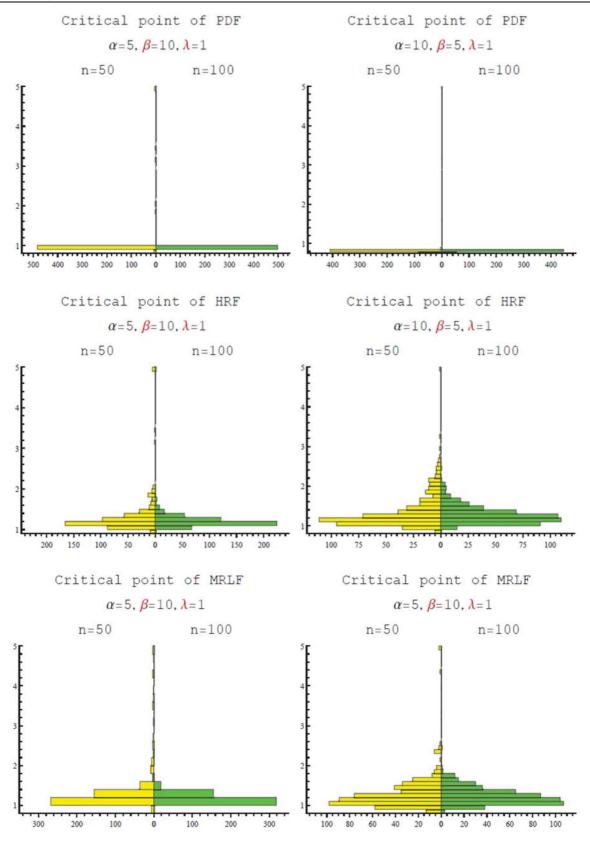
Fig. 5. Histograms of the sampling distributions of the MLEs of the critical point of PDF, HRF and MRLF

Table 4. Descriptive sample statistics

Minimum	1.678	Mean	2.358
Maximum	3.210	Variance	0.075
Mode	2.438	CV	0.116

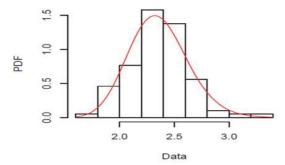Table 5. Parameters estimates, estimated log-likelihood, Kolmogrov-Smirnov and Anderson-Darling tests

Estimated Parameters							
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	Estimated log-likelihood	KS	*p*-value	AD	*p*-value
16.391	3.098	44.632	-10.425	0.082	0.531	0.385	0.863

Table 6. Point and interval estimation for the mean, variance, coefficient of variation, critical points of PDF, HRF and MRLF

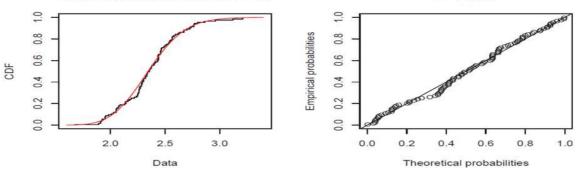|	Asymptotic				Percentile bootstrap			
θ	Estimate	S.E.	95% C.I.	C.I. width	Estimate	S.E.	95% C.I.	C.I. width
μ	2.358	0.028*	(2.304, 2.412)	0.108	2.357	0.028	(2.302, 2.412)	0.110
σ^2	0.073	0.012*	(0.049, 0.097)	0.048	0.073	0.012	(0.052, 0.099)	0.047
cv	0.115	0.009*	(0.096, 0.134)	0.038	0.114	0.009	(0.096, 0.132)	0.036
x_f	2.321	0.039*	(2.244, 2.398)	0.154	2.322	0.039	(2.244, 2.395)	0.151
x_h	4.045	0.913*	(2.255, 5.835)	3.580	4.258	0.913	(3.089, 6.650)	3.561
x_m	3.945	0.950*	(2.083, 5.807)	3.724	4.160	0.950	(2.937, 6.621)	3.684

*: bootstrap standard error

Fig. 6. Diagnostic plots for the fitted PHIW model

Finally, Table 6 compares the estimates of the considered functions of the parameters, their standard error and the corresponding confidence intervals based of asymptotic and percentile bootstrap estimation procedures.
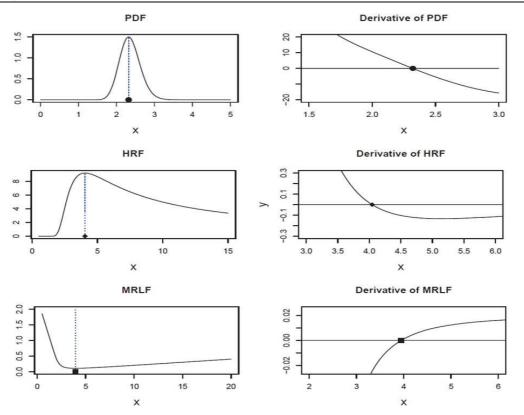
Fig. 7. Location of the percentile bootstrap estimates of the critical points of the PDF, HRF and MRLF of the fitted PHIW model
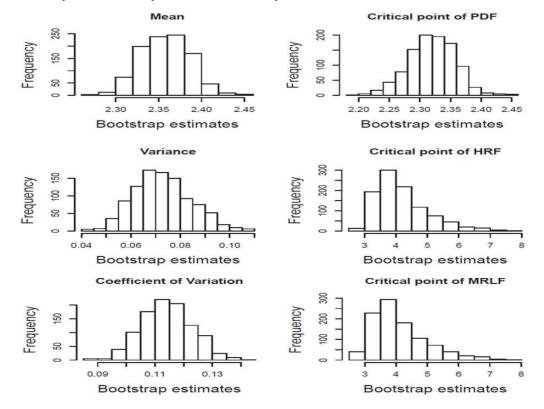

Fig. 8. Histograms of the percentile bootstrap estimates of the mean, variance, coefficient of variation (left column) and critical points of PDF, HRF and MRLF (right column) of fitted PHIW model

Conclusion and Comments

In this study, we have studied the proportional hazard inverse Weibull distribution and the maximum likelihood/bootstrap estimation of various functions of its parameters. It has been shown that the density and hazard rate functions of the proposed model are unimodal for all values of the parameters and its mean residual life function is bathtub shaped under certain condition on two of its parameters. The simulation studies indicate that the asymptotic and bootstrap confidence intervals of various functions of the parameters perform similarly in terms of the coverage probabilities as well as the length of the confidence intervals. We hope that our investigation will provide an alternative way to analyze data which exhibit the characteristics similar to the proposed model.

Author's Contributions

All authors have contributed equally to the work.

Ethics

The authors declare that there is no conflict of interests regarding the publication of this article.

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