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An Integral Two Space-Variables Condition for Parabolic Equations

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Abstract: In this study, an integral two space-variables condition for a class of parabolic equations. The existence and uniqueness of the solution in the functional weighted Sobolev space were proved. The proof is based on two-sided a priori estimates and on the density of the range of the operator generated by the considered problem.

Key words: Integral boundary two space-variables condition, energy inequalities, weighted sobolev spaces

INTRODUCTION

In the domaine $\Omega = \{(x, t) \in (0, 1)\times(0,T), T>0\}$, we consider the equation:

$$Lu = u_t - (a(x,t)u_x)_x = f(x,t) \tag{1}$$

where the function a(x, t) and its derivative are bounded on the interval [0. T]:

$$0 < a_0 < a(x,t) \le a_1$$
$$0 < a_2 \le a_x(x,t) \le a_3$$

To Eq. 1 and 2 we add the initial conditions:

$$\ell u = u(x,0) = \varphi(x), x \in (0,1) \tag{2}$$

The boundary condition Eq. 3:

$$u(0,t) = (1,t)t \in (0,T) \tag{3}$$

And integral condition Eq. 4:

$$\int_0^a u(\xi,t)d\xi + \int_\beta^1 u(\xi,t)d\xi = 0 \quad a > 0, \beta > 0, a < \beta \; a + \beta = 1 \; t \in (0,t) \tag{4}$$

Here, we assumed that the known function φ satisfy the conditions given in (3) and (4), i.e.,

$$\varphi(0) = \varphi(1), \int_0^a \varphi(x)dx + \int_\beta^1 \varphi(x)dx = 0$$

When considering the classical solution of the problem (1)-(4), along with the condition (4) should be fulfilled the conditions:

$$f(0,0) - f(1,0) = 0$$
$$\int_0^\alpha \{a_x(x,0)\varphi'(x) + a(x,0)\varphi''(x)\}dx + \int_\beta^1 \{a_x(x,0)\varphi'(x) + a(x,0)\varphi''(x)\}dx = \int_0^a f(x,0)dx + \int_\beta^1 f(f(x,0)dx$$

Mathematical modeling of different phenomena leads to problems with nonlocal or integral boundary conditions. Such a condition occurs in the case when one measures an averaged value of some parameter inside the domaine. This problems arise in plasma physics, heat conduction, biology and demography, modelling and technological process, see for example (Samarskii, 1980; Hieber and Pruss, 1997; Ewing and Lin, 2003; Shi, 1993; Marhoune, 1990).

Boundary-value problems for parabolic equations with integral boundary condition are investigated by Batten (1963); Bouziani and Benouar (1998); Cannon (1963); (1984); Cannon *et al*. (1987); Ionkin (1977); Kamynin (1964); Field and Komkov (1992); Shi (1993); Marhoune and Bouzit (2005); Marhoune and Hameida (2008); Denche *et al*. (1994); Denche and Marhoune (2001); Marhoune and Latrous (2008); Yurchuk (1986) and many references therein. The problem with integral one space-variable (respectively two space-variables) condition is studied in Fairweather and Saylor (1991) and Denche and Marhoune (2000) (respectively in Marhoune (2007) and Marhoune and Lakhal (2009)).

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The present paper is devoted to the study of a problem with boundary integral two-space-variables condition for a partial differential equation.

We associate to problem (1)-(4) the operator L = (L, l), defined from E into F, where E is the Banach space of functions $u \in L_2(\Omega)$, satisfying (3) and (4), with the finite norm Eq. 5:

$$\|u\|_E^2 = \int_\Omega \theta(x)[|u_t|^2 + |u_{xx}|^2]dxdt + \sup_{0 \le t \le T} \int_0^1 \theta(x)|u_x|^2 dx + \sup_{0 \le t \le T} \int_0^a |u|^2 dx + \sup_{0 \le t \le T} \int_\beta^1 |u|^2 dx \quad (5)$$

And F is the Hilbert space of vector-valued functions F = (f, φ) obtained by completion of the space $L_2(\Omega) \times W_2^2$ (0, 1) with respect to the norm Eq. 6:

$$\|F\|_F^2 = \|(f,\varphi)\|_F^2 = \int_\Omega \theta(x)|f|^2 dxdt + \int_0^1 \theta(x)|\varphi'|dx + \int_0^a |\varphi|^2 dx + \int_\beta^1 |\varphi|^2 dx \quad (6)$$

Where:

$$\theta(x) = \begin{cases} x^2 & 0 < x \le a \\ (1-\beta)^2 & a \le x \le \beta \\ (1-x)^2 & \beta \le x < 1 \end{cases}$$

Using the energy inequalities method proposed in (Yurchuk, 1986), we establish two-sided a priori estimates. Then, we prove that the operator L is a linear homeomorphism between the spaces E and F.

Two-sided a priori estimates:

Theorem1: For any function $u \in E$ we have the a priori estimate Eq. 7:

$$\|Lu\|_F \le k\|u\|_{E'} \quad (7)$$

where the constant k is independent of u.

Proof: Using Eq. 1 and initial conditions (2) we obtain Eq. 8-10:

$$\int_\Omega \theta(x)|Lu|^2 dxdt \le 2\int_\Omega \theta(x)\left[|ut|^2 + 4a_1^2 |u_{xx}|^2\right] dxdt + 4\alpha_3^2 \sup_{0 \le t \le T} \int_0^1 \theta(x)|u_x|^2 dx \quad (8)$$

$$\int_0^1 \theta(x)|\varphi'|^2 dx \le \sup_{0 \le t \le T} \int_0^1 \theta(x)|u_x|^2 dx \quad (9)$$

$$\int_0^a |\varphi|^2 dx \le \sup_{0 \le t \le T} \int_0^a |u|^2 dx, \int_\beta^1 |\varphi|^2 dx \le \sup_{0 \le t \le T} \int_\beta^1 |u|^2 dx, \quad (10)$$

Combining the in equalities (8), (9) and (10), we obtain (7) for $u \in E$

Lemma 2: Marhoune (2007) for $u \in E$ we have Eq. 11:

$$\frac{1}{4}\int_0^a \left|\int_x^a u_t(\xi,t)d\xi\right|^2 dx \le \int_0^a x^2 |u_t|^2 dx \quad (11)$$

Theorem 3: For any function $u \in E$, we have the a priori estimate Eq. 12:

$$\|u\|_E \le k\|Lu\|_{F'} \quad (12)$$

With the constant:

$$k = \frac{4\delta}{\exp(cT)(\delta^2 + 136)} \min\left(\frac{\delta}{2}, \frac{1}{2}, \frac{\delta a_0^2}{16}\right)$$

where c and δ is such that Eq. 13:

$$\delta = \frac{a_0^2 - 4a_1a_3}{2a_1a_0^2} > 0, c < 0 \text{ and} -8ca_0 \ge (40 + \delta a_0)a_3^2 \quad (13)$$

Proof: Define:

$$Mu = \begin{cases} \frac{1}{a}\left(x^2 \frac{\partial u}{\partial t} + 2x\int_x^\alpha \frac{\partial u}{\partial t}(\xi,t)d\xi\right) & 0 < x \le a \\ \frac{(1-\beta)^2}{a}\frac{\partial u}{\partial t} & a \le x \le \beta \\ \frac{1}{a}\left((1-x)^2 \frac{\partial^2 u}{\partial t^2} + 2(1-x)\int_\beta^x \frac{\partial u}{\partial t}(\xi,t)d\xi\right) & \beta \le x < 1 \end{cases}$$

We consider for $u \in E$ the quadratic formula Eq. 14:

$$\mathrm{Re}\int_0^\tau \int_0^1 \exp(-ct)Lu\bar{M}udxdt \quad (14)$$

with the constant c satisfying (14), obtained by multiplying the Eq. 1 by exp (-ct) $\overline{Mu}$, by integrating over Ω^τ, where Ω^τ = (0, 1) × (0, τ), , with $0 \le \tau \le T$, and by taking the real part. Integrating by parts in (14) with the use of boundary conditions (3) and (4), we obtain Eq. 15:

$$\mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)LuM\bar{u}dxdt = \int_0^\tau\int_0^1 \frac{\theta(x)}{a}$$
$$\exp(-ct)\,|\frac{\partial u}{\partial t}|^2\ dxdt + \int_0^\tau\int_0^a \frac{1}{a}\exp(-ct)\left|\int_x^a \frac{\partial u}{\partial t}(\xi,t)\,d\xi\right|^2$$
$$dxdt + \frac{1}{2}\int_0^\tau\int_\beta^1 \frac{1}{a}\exp(-ct)\left|\int_\beta^x \frac{\partial u}{\partial t}(\xi,t)d\xi\right|^2 dxdt$$
$$-\int_0^\tau\int_0^a \frac{a_x}{a^2}\exp(-ct)\left|\int_x^a \frac{\partial u}{\partial t}(\xi,t)d\xi\right|^2$$
$$dxdt + \int_0^\tau\int_\beta^1 (1-x)\frac{a_x}{a^2}\exp(-ct)\left|\int_\beta^x \frac{\partial u}{\partial t}(\xi,t)d\xi\right|^2 dxdt$$
$$+\mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x\partial t}dxdt - \mathrm{Re}$$
$$\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\frac{\partial u}{\partial x}\frac{ax}{a}dxdt - \mathrm{Re}\int_0^\tau\int_0^1 \exp$$
$$(-ct)\theta(x)\frac{\partial u}{\partial x}\frac{ax}{a}\frac{\partial u}{\partial t}dxdt$$
$$-2\,\mathrm{Re}\int_0^\tau\int_0^a x\frac{a_x}{a^2}\exp(-ct)\left(\int_x^a \frac{\partial u}{\partial t}(\xi,t)d\xi\right)u_x dxdt$$
$$-2\,\mathrm{Re}\int_0^\tau\int_0^1 (1-x)\frac{a_x}{a^2}\exp(-ct)\left(\int_\beta^x \frac{\partial u}{\partial t}(\xi,t)d\xi\right)u_x dxdt \quad (15)$$

On the other hand, by using the elementary inequalities, we get Eq. 16:

$$\mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)LuM\bar{u}dxdt \ge \int_0^\tau\int_0^1 \frac{\theta(x)}{a}\exp$$
$$(-ct)\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}$$
$$dxdt - \mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\frac{\partial u}{\partial x}\frac{a_x}{a}\frac{\partial u}{\partial t}dxdt \quad (16)$$
$$-\int_0^\tau\int_0^a x\frac{a_x}{a^2}\exp(-ct)\left|\int_x^a \frac{\partial u}{\partial t}(\xi,t)d\xi\right|^2 dxdt$$
$$-\int_0^\tau\int_0^a x^2\frac{a_x}{a^2}\exp(-ct)\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_0^\tau\int_\beta^1 (1-x)^2\frac{a_x}{a^2}$$
$$\exp(-ct)\left|\frac{\partial u}{\partial x}\right|^2 dxdt + 2\,\mathrm{Re}\int_0^\tau\int_0^a \exp(-ct)\frac{\partial u}{\partial t}$$
$$udxdt + 2\,\mathrm{Re}\int_0^\tau\int_\beta^1 \exp(-ct)\frac{\partial u}{\partial t}udxdt$$

Again, using the elementary inequalities and lemma 1 we obtain:

$$\mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)LuM\bar{u}dxdt \ge \delta\int_0^\tau\int_0^1 \exp(-ct)$$
$$\theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)\,\theta(x)\frac{\partial u}{\partial x}\frac{\partial^2}{\partial_x}$$
$$dxdt + 2\,\mathrm{Re}\int_0^\tau\int_0^a \exp(-ct)\frac{\partial u}{\partial t}udxdt \quad (17)$$
$$2\,\mathrm{Re}\int_0^\tau\int_\beta^1 \exp(-ct)\frac{\partial u}{\partial t}udxdt -$$
$$\frac{5a_3^2}{2a_0}\int_0^\tau\int_0^1 \exp(-ct)\,|\frac{\partial u}{\partial x}|^2\ dxdt$$

Integrating by parts the second, third and forth terms of the right-hand side of the inequality (17) and taking into account the initial condition (2) and the condition (13) give Eq. 18:

$$\mathrm{Re}\int_0^\tau\int_0^1 \exp(-ct)LuM\bar{u}dxdt + \int_0^a|\varphi|^2 dx + \int_\beta^1|\varphi| dx.$$
$$+\frac{1}{2}\int_0^1 \theta(x)|\varphi'|^2 dx \ge \delta\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dxdt$$
$$+\frac{1}{2}\int_0^1 \theta(x)\exp(-ct)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx + \int_0^a \exp \quad (18)$$
$$(-c\tau)|u(x,\tau)|^2 dx + \int_\beta^1 \exp(-c\tau)|u(x,\tau)|^2 dx$$

By using the elementary inequalities on the first integral in the left-hand side of (18) we obtain:

$$\frac{3\delta}{2}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dxdt +$$
$$\frac{1}{2}\int_0^1 \theta(x)\exp(-c\tau)|u_x(x,\tau)|^2 dx$$
$$+\int_0^a \exp(-c\tau)|u(x,\tau)|^2 dx + \int_\beta^1 \exp(-c\tau)|u(x,\tau)|^2 dx \quad (19)$$
$$\le \frac{34}{\delta}\int_0^\tau\int_0^1 \theta(x)\exp(-ct)\theta(x)|Lu|^2 dxdt + \frac{1}{2}$$
$$\int_0^1 \theta(x)|\varphi'|^2 dx + \int_0^a|\varphi|^2 dx + \int_\beta^1|\varphi|^2 dx$$

Now, from Eq. 1 we have:

$$\int_0^\tau\int_0^1 \frac{\delta a^2}{16}\theta(x)\exp(-ct)\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dxdt$$
$$\le \frac{\delta}{4}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)|Lu|^2 dxdt$$
$$+\frac{\delta}{4}\int_0^\tau\int_0^1 \exp(-ct)\theta(x)|u_t|^2 dxdt + 2 \quad (20)$$
$$\int_0^\tau\int_0^1 \frac{\delta a_x^2}{16}\exp(-ct)\theta(x)\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dxdt$$

Combining inequalities (19), (20) we get Eq. 21:

$$\frac{\delta^2+\exp(-cT)}{4\delta}\int_\Omega \theta(x)|Lu|^2 dxdt + \frac{1}{2}\int_0^1 \theta(x)|\varphi'|^2$$
$$dx + \int_0^a|\varphi|^2 dx + \int_\beta^1|\varphi|^2 dx \ge \quad (21)$$
$$\left(\begin{array}{c}\theta(x)|u_t|^2 dxdt + \frac{1}{2}\int_0^1 \theta(x)|u_x(x,\tau)|^2 \\ \frac{\delta}{2}\int_0^\tau\int_0^1 dx + \int_0^a|u(x,\tau)|^2 dx + \int_\beta^1|u(x,\tau)|^2 \\ dx + \frac{\delta}{2}\int_0^\tau\int_0^1 \theta(x)|u_{xx}|^2 dxdt\end{array}\right)$$

As the left-hand side of (21) is independent of τ, by remplacing the right-hand side by its upper bound with respect to τ in the interval [0, T], we obtain the desired inequality.

Solvability of the problem: The proof of existence of solution is based on the following lemma.

Lemma 4: Let: D_0 (L) = {u∈ E: φu=0} If for u∈ D_0 and some ω∈ L_2 (Ω), we have Eq. 22:

$$\int_\Omega \phi(x) Lu\varpi dxdt = 0 \tag{22}$$

where

$$\phi(x) = \begin{cases} x & 0 < v \le a \\ (1-\beta) & a \le x \le \beta \\ (1-x) & \beta \le x < 1 \end{cases}$$

Then, ω = 0.

Proof: From (22) we have:

$$\int_\Omega \phi(x) u_t \varpi dxdt = \int_\Omega \phi(x)(a(x,t)u_x)_x \varpi dxdt \tag{23}$$

Now, for given ω, we introduce the function:

$$v(x,t) = \begin{cases} \omega - \int_x^a \frac{\omega(\xi,t)}{\xi} d\xi & 0 < x \le a \\ \omega & a \le x \le \beta \\ \omega - \int_\beta^x \frac{\omega(\xi,t)}{1-\xi} d\xi & \beta \le x < 1 \end{cases}$$

Integrating by parts with respect to ξ, we obtain:

$$Nv = \phi(x)\omega = \begin{cases} xv + \int_x^a v(\xi,t)d\xi & 0 < x \le a \\ (1-\beta)v & a \le x \le \beta \\ (1-x)v + \int_\beta^x v(\xi,t)d\xi & \beta \le x < 1 \end{cases}$$

Which implies that Eq. 24:

$$\int_x^a v(\xi,t)d\xi = \int_\beta^1 v(\xi,t)d\xi = 0 \tag{24}$$

Then, from equality (23) we obtain Eq. 25:

$$-\int_\Omega u_t \bar{N} u dxdt = \int_\Omega A(t)u\bar{v}dxdt \tag{25}$$

Where:

$$A(t)u = -(\phi(x)a(x,t)u_x)_x$$

If we introduce the smoothing operators with respect to t (Yurchuk, 1986; Marhoune and Lakhal, 2009), $J_\varepsilon^{-1} = (I + \varepsilon \frac{2}{\partial t})^{-1}$ and $(J_\varepsilon^{-1})^*$, then these operators provide the solutions of the respective problems Eq. 26:

$$\begin{aligned} &\varepsilon(g_\varepsilon)_t(t) + g_\varepsilon(t) = g(t) \\ &g_\varepsilon(t)\big|_{t=0} = 0 \end{aligned} \tag{26}$$

And Eq. 27:

$$\begin{aligned} &-\varepsilon(g_\varepsilon^*)_t(t) + g_\varepsilon^*(t) = g(t) \\ &g_\varepsilon(t)\big|_{|t=T} = 0 \end{aligned} \tag{27}$$

And also have the following properties : for any g ∈ L_2 (0, T), the functions $g_\varepsilon = (J_\varepsilon^{-1})g$ and $g_\varepsilon^* = (J_\varepsilon^{-1})g$ are in $W_2^1(0,T)$ such that g_ε |t = 0 = 0 and $g_\varepsilon^* \mid t = T = 0$. Moreover, J_ε^{-1} commutes with $\frac{\partial}{\partial t}$, so $\int_0^T |g_\varepsilon - g|^2 dt \to 0$ and $\int_0^T |g_\varepsilon - g|^2 dt \to 0$ for ε→0. Putting $\int_0^t \exp(c\tau)u_\varepsilon^*(x,\tau)d\tau$ in (25), where the constant c satisfies $ca_0 - a_3 - \frac{\varepsilon a_3^2}{2a_0} \ge 0$ and using (27), we obtain Eq. 28:

$$-\int_\Omega \begin{aligned} &\exp(ct)u_\varepsilon^* \bar{N} u dxdt = \\ &+\int_\Omega A(t)u\exp(-ct)\bar{u}_t dxdt - \varepsilon\int_\Omega A(t)u(\bar{u}_\varepsilon^*)_t \end{aligned} dxdt \tag{28}$$

Integrating by parts each term in the left-hand side of (28) and taking the real parts yield Eq. 29 and 30:

$$\begin{aligned} &2\,\mathrm{Re}\int_\Omega A(t)u\exp(-ct)\bar{u} dxdt = \\ &\int_0^1 a(x,t)\exp(-ct)\phi(x)\left|u_t(x,T)\right|^2 dx \\ &+\int_\Omega \exp(-ct)\phi(x)(ca(x,t) - a_t(x,t))\left|u_x\right|^2 dxdt \end{aligned} \tag{29}$$

$$\begin{aligned} &\mathrm{Re}\left(-\varepsilon\int_\Omega A(t)u\exp(-ct)\overline{(v_\varepsilon^*)}_t dxdt\right) \\ &= \mathrm{Re}\left(\varepsilon\int_\Omega a_t(x,t)\phi(x)u_x\overline{(v_\varepsilon^*)}_x dxdt\right) \\ &+\int_\Omega a(x,t)\exp(-ct)\phi(x)\left|\overline{(v_\varepsilon^*)}\right|^2 dxdt \end{aligned} \tag{30}$$

Using ε-inequalitiesweobtain Eq. 31:

$$\begin{aligned} &\operatorname{Re}\left(-\varepsilon\int_\Omega A(t)u\exp(-ct)\overline{(v_\varepsilon^*)_t}\,dxdt\right) \\ &\geq \int_\Omega \exp(-ct)\phi(x)\left|a_t(x,t)\right|^2\left|u_x\right|^2 dxdt \end{aligned} \tag{31}$$

Combining (29) and (31) we get Eq. 32:

$$\begin{aligned} &-\operatorname{Re}\left(\int_\Omega \exp(ct)v_\varepsilon^*\overline{N}vdxdt\right) \geq \\ &\int_\Omega \exp(-ct)\phi(x)\left(ca_0 - a_3 - \frac{\varepsilon a_0^2}{2a_0}\right)\left|u_x\right|^2 dxdt \geq 0 \end{aligned} \tag{32}$$

Now, using (32), we have:

$$\operatorname{Re}\int_\Omega \exp(ct)v_\varepsilon^*\overline{N}vdxdt \leq 0$$

Then, for ε→0 we obtain:

$$\operatorname{Re}\int_\Omega \exp(ct)v\overline{N}vdxdt = \int_\Omega \exp(ct)\phi(x)\,|\,v\,|^2\,dxdt \leq 0$$

We conclude that v = 0; hence, ω = 0, which ends the proof of the lemma.

Theorem 5: The range R (L) of L coincides with F.
Proof: Since F is a Hilbert space, we have R (L) = F if and only if the relation Eq. 33:

$$\begin{aligned} &\int_\Omega \theta(x)Lu\overline{f}dxdt + \\ &\int_0^1 \theta(x)\frac{dlu}{dx}\frac{d\overline{\varphi}}{dx}dx + \int_0^a lu\overline{\varphi}dx + \int_\beta^1 lu\overline{\varphi}dx = 0 \end{aligned} \tag{33}$$

For arbitrary $u \in E$ and $(f\varphi) \in F$, implies that f = 0, and φ = 0. Putting $u \in D_0$ (L) in (33), we conclude from the lemma 3 that Ψf = 0, then f = 0 where:

$$\psi f = \begin{cases} xf & 0 < x \leq a \\ (1-\beta)f & a \leq x \leq \beta \\ (1-x)f & \beta \leq x < 1 \end{cases}$$

Taking $u \in E$ in (33) yield:

$$\int_0^1 \theta(x)\frac{dlu}{dx}\frac{d\overline{\varphi}}{dx}dx + \int_0^a lu\overline{\varphi}dx + \int_\beta^1 lu\overline{\varphi}dx = 0$$

The range of the trace operator l is everywhere dense in Hilbert space with the norm:

$$\left[\int_0^1 \theta(x)\left|\frac{d\varphi}{dx}\right|^2 + \int_0^a |\varphi|^2\,dx + \int_\beta^1 |\varphi|^2\,dx\right]^{\frac{1}{2}};\ \text{hence}, \varphi = 0$$

CONCLUSION

From estimates (7) and (11) it follows that the operator L: E→F is continuous and its range is closed in F. Therefore, the inverse operator L^{-1} exists and is continuous from the closed subspace R (L) onto E, which means that L is a homeomorphism from E onto R (L). The theorem 5 chow that R (L) = F. So the existence and uniqueness of the solution of the problem is proved.

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