

Exterior Differential Systems for Higher Order Partial Differential Equations

Paul Bracken

Department of Mathematics, University of Texas, Edinburg, TX 78541-2999

Abstract: The theory of exterior differential systems is applied to study integrability of a set of related partial differential equations expressed in the terms of differential forms using Cartan's method. The Camassa-Holm equation and the Degasperis-Procesi equations are special cases that are described by these exterior differential systems. Some conservation laws are obtained in the cases of the more relevant equations. A closed differential ideal is constructed for each case studied.

Key words: Differential form, partial differential equation, integrability, Camassa-Holm

INTRODUCTION

It has been known for a while that the simplest nonlinear evolution equations which have solitary wave solutions or solitons are known to have an infinite number of conservation laws. This in turn is related to the concept of integrability of a particular partial differential equation or system of such equations (Estabrook and Wahlquist, 1975). There are many implications and applications of the idea of integrability. For example, a Lax pair can usually be determined and a Backlund or auto-Backlund transformation might be written down.

It was shown by Wahlquist and Estabrook (1975; 1976) that by applying results from the theory of exterior differential systems, as well as prolongation techniques, it is possible to determine integrability of a related partial differential system. The use of prolongation results can often lead to a Lax pair or Backlund transformation for the given nonlinear system.

The equation which was studied by Wahlquist and Estabrook is the classical Korteweg-de Vries equation, which has been known to describe the long time evolution of finite amplitude waves. The equation has many applications which extend beyond the original applications to solitary surface waves. It seems to be that nonlinear dispersion can act to compatify solitary waves and generate solitons with a finite wavelength. Recently a generalized Korteweg-de Vries equation (Rosenau and Hyman, 1993; Sivers and Takens, 1988; Bracken, 2005) has been studied. The symmetry group has been determined for this equation and several classes of solutions have been obtained, which include both soliton solutions and solutions which have compact support (Pikorsky and Rosenau, 2006). The

type of approach which was begun by Wahlquist and Estabrook has been applied to this generalized equation as far as determining some prolongations is concerned (Bracken, 2007).

It is the intention here to review some of the mathematical background which will let us study some interrelated equations which have been of interest recently. These ideas are finally applied to a class of equations that includes the Camassa-Holm and Degasperis-Procesi equations. These equations are of the form:

$$(u-u_{xx})_t + u(u-u_{xx})_x + \beta(u-u_{xx})u_x = 0 \quad (1)$$

for specific choices of the constant β .

An exterior differential system which reproduces the given equation on the transverse manifold is developed for each case. The derivatives of the forms in this set are shown to be expressible in terms of the same forms, so the integrability of each equation is established. Finally, conservation laws for the two equations will be written down developed from the original set of one-forms.

Differential systems and their equations: Consider the space $M = \mathbb{R}^n(x, t, u, p, q, \dots)$ in which there is defined a closed exterior differential system which will be written:

$$\alpha_1 = 0, \dots, \alpha_m = 0 \quad (2)$$

Let I be the ideal generated by the set $\{\alpha_i\}_{i=1}^m$ in (2). This is given explicitly as:

$$I = \left\{ \omega = \sum_{i=1}^m \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda(M) \right\} \quad (3)$$

If the ideal in (3) is closed, it follows that $dI \subset I$ and so we may say that (2) is integrable. It is important to stress that system (2) is chosen in such a way that the solutions $u = u(x, t)$ of a specific equation:

$$u_t = F(x, t, u, u_x, u_{xx}, \dots) \quad (4)$$

correspond with the two-dimensional integral manifolds of (2). These are the integral manifolds given by sections S of the projection:

$$\pi: M \rightarrow \mathbb{R}^2, \quad \pi(x, t, u, p, q, \dots) = (x, t) \quad (5)$$

These sections S are given by a mapping:

$$S: \mathbb{R}^2 \rightarrow M, \quad S(x, t) = (x, t, u(x, t), p(x, t), q(x, t), \dots) \quad (6)$$

Due to the required transversality, we have $dx \wedge dt|_S = \pi^*(dx \wedge dt) \neq 0$.

Several exterior differential systems will be discussed here. It will be seen that we end with a particular differential system which leads to some partial differential equations of interest, in particular, an equation of the form (1):

- Let us introduce the differential system defined to be:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p \, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q \, dx \wedge dt \\ \alpha_3 &= du \wedge dx - dq \wedge dx + dq \wedge dt + (u - q) \, dx \wedge dt \end{aligned} \quad (7)$$

The exterior derivatives of the β_i can be calculated and we obtain:

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = \frac{1}{q} \alpha_2 \wedge dp \\ d\alpha_2 &= -dq \wedge dx \wedge dt = \alpha_3 \wedge dx \\ d\alpha_3 &= du \wedge dx \wedge dt - dq \wedge dx \wedge dt = \alpha_3 \wedge dt \end{aligned} \quad (8)$$

Therefore, it can be seen that all of these exterior derivatives vanish modulo $\{\alpha_j\}_{j=1}^3$. Any regular two-dimensional solution manifold in the five-dimensional space $S_2 = \{u(x, t), u_x = p(x, t), p_x(x, t) = q(x, t)\}$ satisfying a specific partial differential equation of the form (4) will annul this set of forms. The exact form of this equation which corresponds to (5) can be found explicitly by sectioning the forms into the solution manifold. On

S_2 , we must have $d_u = u_x \, d_x + u_t \, dt$, and similarly for both p and q . It follows that:

$$\begin{aligned} 0 &= \alpha_1|_S = (u_x - p) \, dx \wedge dt \\ 0 &= \alpha_2|_S = (p_x - q) \, dx \wedge dt \\ 0 &= \alpha_3|_S = (u_t - q_t - (u - q) - q_x) \, dt \wedge dx \end{aligned} \quad (9)$$

Therefore, $p = u_x$ and $q = u_{xx}$ with the equation given by:

$$(u - u_{xx})t - (u - u_{xx}) - u_{xxx} = 0 \quad (10)$$

This is the specific equation whose integrability is implied by system (7)

- Consider the differential system:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p \, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q \, dx \wedge dt \\ \alpha_3 &= du \wedge dx - dq \wedge dx + du \wedge dt - dq \wedge dt + (u - q) \, dx \wedge dt \end{aligned} \quad (11)$$

Differentiating each of these, we obtain:

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2 \\ d\alpha_2 &= -dq \wedge dx \wedge dt = dx \wedge (-\alpha_3 + \alpha_1) \\ d\alpha_3 &= -dx \wedge \alpha_3 \end{aligned} \quad (12)$$

Upon sectioning these forms, we again obtain that $p = u_x$, $q = u_{xx}$ and the equation which pertains to (11) arises from the section $\alpha_3|_S = 0$ and is given by:

$$(u - u_{xx})t - (u - u_{xx})_x - (u - u_{xx}) = 0 \quad (13)$$

The final two cases which will be introduced include equations which are being actively studied at the moment

- Let β be a real, nonzero constant. Define the exterior differential system as the collection of two-forms

$$\begin{aligned} \alpha_1 &= du \wedge dt - p \, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q \, dx \wedge dt \\ \alpha_3 &= -du \wedge dx + dq \wedge dx - \beta \, du \, dq \wedge dt + \beta(2u - q) \, du \wedge dt \end{aligned} \quad (14)$$

The exterior derivatives of the α_j are given by:

$$\begin{aligned} d\alpha_1 &= dx \wedge dp \wedge dt = dx \wedge \alpha_2 \\ d\alpha_2 &= dx \wedge dq \wedge dt = \\ &\frac{1}{\beta u} dx \wedge (-\alpha_3 + \beta u \alpha_1 + \beta(u-q)\alpha_1) \end{aligned} \quad (15)$$

$$d\alpha_3 = 0$$

These vanish modulo the set of α_j from (16) and carrying out the sectioning, the equation is obtained from the restriction of α_3 :

$$\alpha_3|_S = (u_t - q_t + \beta u u_x - \beta u q_x + \beta(u-q)u_x) dx \wedge dt$$

Using $p = u_x$ and $q = u_{xx}$ from sectioning α_1 and α_2 , we have finally:

$$(u - u_{xx})_t + \beta(u - u_{xx})_x = 0 \quad (16)$$

- The following system leads to an important class of partial differential equations which are of much current interest. The Camassa-Holm and Degasperis-Procesi equations are two particular examples which appear. Define the system of forms:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q dx \wedge dt \\ \alpha_3 &= -du \wedge dx + dq \wedge dx - u dq \wedge dt + \\ &\quad u du \wedge dt + \beta(u-q)du \wedge dt \end{aligned} \quad (17)$$

The quantity β which appears in α_3 is a real, nonzero constant. Differentiating each of these, we have:

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2 \\ d\alpha_2 &= 1 dx \wedge (-\alpha_3 + u((1+\beta)u - q)\alpha_1) \\ d\alpha_3 &= (1-\beta)dq \wedge du \wedge dt \\ &= (1-\beta) dq \wedge \alpha_1 + (1-\beta)p dq \wedge dx \wedge dt \\ &= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p(\alpha_3 + du \wedge dx) \wedge dt \\ &= (1-\beta)dq \wedge \alpha_1 + (1-\beta) p dt \wedge \alpha_3 \\ &\quad - (1-\beta)p dx \wedge du \wedge dt \\ &= (1-\beta)[dq \wedge \alpha_1 + p dt \wedge \alpha_3 - p dx \wedge \alpha_1] \end{aligned} \quad (18)$$

All of the details for calculating $d\alpha_3$ have been shown here. Clearly all of the $d\alpha_j$ vanish modulo the set of α_j . Sectioning α_1 and α_2 gives $p = u_x$ and $q = u_{xx}$ as in the other cases and the equation results from evaluating the section as follows:

$$0 = \alpha_3|_S = (u_t - q_t + u(u_x - q_x) + \beta(u-q)u_x) dx \wedge dt$$

These results imply the partial differential equation:

$$(u-q)_t + u(u-q)_x + \beta(u-q)u_x = 0 \quad (19)$$

Substituting $q = u_{xx}$ into (19), an equation of the form (1) is obtained:

$$(u - u_{xx})_t + u(u - u_{xx})_x + \beta(u - u_{xx})u_x = 0 \quad (20)$$

If we introduce $\rho = u - u_{xx}$, then for the case $\beta = 3$, Eq. 20 becomes the Degasperis-Procesi equation:

$$\rho_t + \rho_x u + 3\rho u_x = 0, \rho = u - u_{xx} \quad (21)$$

Finally, for $\beta = 2$, (20) is the Camassa-Holm equation:

$$\rho_t + \rho_x u + 2\rho u_x = 0, \rho = u - u_{xx} \quad (22)$$

Conservation laws: Consider now the related subject of conservation laws for these systems. Conservation laws associated with the equations of the last two cases correspond to the existence of exact two-forms contained in the ring of the forms $\{\alpha_j\}$. Let us suppose that we can find a set of functions $f_i(x, t, u, p, q)$ such that the two-form:

$$\sigma = f_1 \alpha_1 + f_2 \alpha_2 + f_3 \alpha_3 \quad (23)$$

satisfies $d\sigma = 0$, the condition for exactness. This is the integrability condition for the existence of a one-form ω such that:

$$\sigma = d\omega \quad (24)$$

which conversely implies that $d\sigma = 0$. Only the final two equations which were discussed in Section 2 will be considered now.

For case (iii), consider the one-form σ given by:

$$\sigma = \alpha_3 \quad (25)$$

It has already been shown by differentiating σ that $d\sigma = 0$. In fact, β can be derived from a single one-form. Let ω be defined to be:

$$\omega = (q-u) dx + \beta u(u-q) dt \quad (26)$$

By working out the exterior derivative of ω , we obtain:

$$d\omega = -du \wedge dx + dq \wedge dx + \beta u du \wedge dt - \beta u dq \wedge dt + \beta(u - q) du \wedge dt$$

This is exactly the form σ in (25).

With regard to case (iv), with the $\{\alpha_i\}$ given by (17), we can similarly define a form σ . To do calculations here, the exterior derivatives of the forms in (17) can be simplified to read:

$$d\alpha_1 = -dp \wedge dx \wedge dt, \quad d\alpha_2 = -dq \wedge dx \wedge dt, \quad d\alpha_3 = (1 - \beta) dq \wedge du \wedge dt$$

In this case, consider the one-form β which is defined to be:

$$\sigma = \alpha_3 - (1 - \beta)q\alpha_1 + (1 - \beta)p\alpha_2 \tag{27}$$

Calculating the exterior derivatives of σ , it is found that it vanishes:

$$d\sigma = (1 - \beta) dq \wedge du \wedge dt - (1 - \beta)dq \wedge du \wedge dt + (1 - \beta)pdq \wedge dx \wedge dt + (1 - \beta)qdp \wedge dx \wedge dt - (1 - \beta)qdp \wedge dx \wedge dt - (1 - \beta)pdq \wedge dx \wedge dt = 0$$

Now it can be shown that β can be derived from a one-form, namely ω defined by:

$$\omega = (q - u)dx + \frac{1}{2}(u^2 - 2uq + \beta u^2 + p^2)dt \tag{28}$$

Differentiating ω , it is obtained that:

$$d\omega = -du \wedge dx + dq \wedge dx + u du \wedge dt - u dq \wedge dt + \beta u du \wedge dt - q du \wedge dt + (1 - \beta)pdp \wedge dt \tag{29}$$

This is precisely the two-form σ given in (27).

The associated conservation law results from an application of Stokes theorem, which is:

$$\oint_{M_1} \omega = \oint_{M_2} d\omega \tag{30}$$

This has been written for any simply-connected, two-dimensional manifold M_2 with closed onedimensional boundary M_1 . The notation implies that ω and $d\omega$ are to be evaluated on their respective manifolds.

Considering ω again, we can of course add to this ω any exact one-form dv , where v is an arbitrary, scalar function. Thus ω can be taken to be:

$$\omega = dv + (q - u)dx + \frac{1}{2}(u^2 - 2uq + \beta u^2 + p^2) dt \tag{31}$$

such that $\sigma = d\omega$. Now v may be regarded simply as a coordinate in an extended six-dimensional space of variables $\{x, t, u, p, q, v\}$ and the one-form ω may be included with the original set of forms. Since $d\omega$ is known to be in the ring of the original set, the new set of forms remains a closed ideal.

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