Journal of Mathematics and Statistics 4 (1): 58-59, 2008
ISSN 1549-3644

A New Approach for Solving Second Order Ordinary Differential Equations

Laith K. AL-Hwawcha and Namh A. Abid
Al-Zaytoonah University, Amman, Jordan

Abstract: A new approach is presented to solve second order linear differential equations with variable coefficients and some illustrative examples are given.

Key words: Second order equations, general solution, homogeneous and nonhomogeneous equations

INTRODUCTION

Consider the second order linear ordinary differential equation

$$y'' + P(x)\, y' + Q(x)\, y = G(x) \qquad (1)$$

where, P, Q and G are continuous functions. It is known that the power series method is a powerful method for solving Eq.(1). However, this method needs a lot of time, space and high concentration during calculations. In this research, we present a new approach which can be used to a wide class of equations either to find a general solution to the associated homogeneous equation or to find a particular solution to Eq.(1) without requiring the general solution or any solution of the associated homogeneous equation as most methods require. For more details, see[1].

MAIN RESULTS

In this section we introduce our main results.

Theorem 1: Consider the equation

$$y'' + P(x)\, y' + Q(x)\, y = 0 \qquad (2)$$

If $v(x) = y'(x) + \beta(x)y(x)$, where $\beta(x)$ is a solution of the Riccati equation $\beta'(x) = Q(x) - P(x)\,\beta(x) + \beta^2(x)$,
then,

$$y(x) = e^{-\int \beta(x)dx} \int e^{\int (2\beta(x) - P(x))dx} dx \qquad (3)$$

is a solution of Eq.(2).

Proof: It is easy to show that $v' = (\beta(x) - P(x))\, v$, where Riccati equation has been used and $v(x) = e^{\int (\beta(x) - P(x))dx}$, then the result is achieved.

Note: It is known that the substitution $v(x) = \frac{-y'}{y}$ transfers Eq. (2) to a Riccati equation and $y = e^{-\int v(x)dx}$ is a solution of the equation. This result is included in the theorem (1) and the formula (3) really gives a second linearly independent solution to Eq. (2) and therefore the general solution is constructed. These facts are illustrated in the following example.

Example 1: Find a general solution of the equation

$$x\, y'' - (1+x)\, y' + y = 0 \qquad (4)$$

Solution: Here, $P(x) = \frac{-(1+x)}{x}$, $Q(x) = \frac{1}{x}$, so the Riccati equation is

$$\beta'(x) = \frac{1}{x} + \left(\frac{1+x}{x}\right)\beta(x) + \beta^2(x)$$

and $\beta(x) = -1$ is a solution of the equation, and then $y_1(x) = e^{\int dx} = e^x$ is a solution of the equation. Thus

$$\begin{aligned} y_2(x) &= e^{\int dx} \int e^{\int \left(-2 + \frac{1+x}{x}\right)dx} dx \\ &= -x - 1. \end{aligned}$$

Hence the general solution is

$$y(x) = c_1 e^x + c_2(x+1)\,.$$

By using the same technique, naturally one can get the following result, which can be used to find a particular solution of Eq. (1). In particular, this procedure can be used easily to find a particular solution of second order ordinary differential equations

Corresponding Author: Laith K. AL-Hwawcha, Al-Zaytoonah University, Amman, Jordan

with constants coefficients and for Cauchy- Euler equation because the associated Riccati equation is solvable.

Theorem 2: Consider the equation

$$y''+P(x)\,y'+Q(x)\,y = G(x) \quad (5)$$

If $v(x) = y'(x)+\beta(x)y(x)$, where $\beta(x)$ is a solution of the Riccati equation

$$\beta'(x) = Q(x) - P(x)\,\beta(x) + \beta^2(x),$$

then

$$y(x) = e^{-\int \beta(x)dx}\int (e^{\int (2\beta(x)-P(x))dx}\int G(x)e^{-\int (\beta(x)-P(x))dx}dx)dx$$

is a solution of Eq. (5).

Example 2: Find a particular solution of the equation

$$x^2y'' + 3xy' + y = x^2\ln x, x > 0 \quad (6)$$

Solution: Here, $P(x) = \frac{3}{x}$ and $Q(x) = \frac{1}{x^2}$, so the Riccati equation is given by:

$$\beta'(x) = \frac{1}{x^2} - \frac{3}{x}\beta(x)+\beta^2(x),$$

and $\beta(x) = \frac{1}{x}$ is a solution of the equation. Thus

$$y_p(x) = e^{-\int \frac{1}{x}dx}\int (e^{-\int \frac{1}{x}dx}\int \ln(x)e^{2\int \frac{1}{x}dx}dx)dx$$
$$= \frac{1}{9}x^2(\ln(x) - \frac{2}{3})$$

is a particular solution of the given equation.

CONCLUSION

In this research we introduce a new approach for solving second order ordinary differential equations, and it seems an easier way to teach these equations than the usual ones.

REFERENCES

1. Boyce, W.E. and R.C. DiPrima, 2000. Elementary Differential Equations and Boundary Value Problems. John Wiley and Sons, Inc.