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An Operator Defined by Convolution Involving the Polylogarithms Functions

K. Al Shaqsi and M. Darus
School of Mathematical Sciences, Faculty of Science and Technology
University Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

Abstract: We define an operator on the class $\mathcal{A}$ of analytic functions in the unit disk $\mathbb{U} = \{z :|z|< 1\}$ involving the polylogarithms functions and introduce certain new subclasses of $\mathcal{A}$ using this operator. Some inclusion results, covering theorem, coefficients inequalities, and several other interesting properties of these classes are obtained.
Key words: Analytic functions, univalent functions, polylogarithms functions, derivative operato

INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the unit disk $\mathbb{U} = \{z :|z|< 1\}$. For functions f given by (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, let $(f * g)(z)$ denote the Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. And for the functions $f(z)$ and $g(z)$ in $\mathcal{A}$, we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a Schwarz function w in $\mathcal{A}$ with $|w(z)|< 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in $\mathbb{U}$.

For $f \in \mathcal{A}$, Sălăgean[9] has introduced the following operator called the Sălăgean operator:

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D(D^{n-1} f(z)), \quad (n \in \mathbb{N}).$$

Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,$$
$$(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Let $f \in \mathcal{A}$. Denote by $D^\lambda : \mathcal{A} \to \mathcal{A}$, the operator defined by:

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1).$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$ and

$$D^\delta f(z) = \frac{z(z^{\delta-1} f(z))^{(\delta)}}{\delta!}, \quad (\delta \in \mathbb{N}_0).$$

Note that $D^\delta f(z) = z + \sum_{k=2}^{\infty} C(\delta, k) a_k z^k$, where $C(\delta, k) = \binom{k+\delta-1}{\delta}$ and $\delta \in \mathbb{N}_0$.

The operator $D^\delta f$ is called the Ruscheweyh derivative operator[8].

Finally, let P denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ analytic in $\mathbb{U}$ which satisfy the condition $\mathrm{Re}\{p(z)\} > 0$.

We recall here the definition of the well-known generalization of the polylogarithm function $G(n; z)$ given by

$$G(n; z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (n \in \square,\ z \in \mathbb{U}). \qquad (2)$$

We note that $G(-1; z) = z/(1-z)^2$ is Koebe function. For more about polylogarithms in theory of univalent functions see Ponnusamy and Sabapathy[7] and Ponnusamy[6].

Corresponding Author: M. Darus, Faculty of Science and Technology, School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

We now introduce a function $(G(n;\, z))^{(-1)}$ given by

$$G(n;\, z) \,*\, (G(n;\, z))^{(-1)} = \frac{z}{(1-z)^{\lambda+1}},$$

$$(\lambda > -1,\ n \in \mathbb{N}) \tag{3}$$

and obtain the following linear operator

$$\mathfrak{D}_\lambda^n f(z) = (G(n;\, z))^{(-1)} * f(z). \tag{4}$$

Now we find the explicit form of the function $(G(n;\, z))^{(-1)}$. It is well known that for $\lambda > -1$ we have:

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k\,!} z^{k+1} \quad (z \in \mathbb{U}). \tag{5}$$

Putting (3) and (5) in (4), we get:

$$\sum_{k=1}^{\infty} \frac{1}{k^n} z^k * (G(n;\, z))^{(-1)} = \sum_{k=1}^{\infty} \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k .$$

Therefore the function $(G(n;\, z))^{(-1)}$ has the following form

$$(G(n;\, z))^{(-1)} = \sum_{k=1}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k \quad (z \in \mathbb{U}).$$

For $n, \lambda \in \mathbb{N}_0$, we note that

$$\mathfrak{D}_\lambda^n f(z) = z + \sum_{k=2}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k \quad (z \in \mathbb{U}). \tag{6}$$

Note that $\mathfrak{D}_0^n \equiv D^n$ and $\mathfrak{D}_\lambda^0 \equiv D^\delta$ which are Sălăgean and Ruscheweyh derivative operators, respectively[9,8]. It is clear that the operator $\mathfrak{D}_\lambda^n$ included two known derivative operators. Also note that $\mathfrak{D}_0^0 f(z) = f(z)$ and $\mathfrak{D}_0^1 f(z) = \mathfrak{D}_1^0 f(z) = zf'(z)$.

Definition 1: Let $K_\lambda^n(\phi(z))$ be the class of functions $f \in \mathcal{A}$ for which

$$\frac{z(\mathfrak{D}_\lambda^n f(z))'}{\mathfrak{D}_\lambda^n f(z)} \prec \phi(z), \tag{7}$$

$$(n, \lambda \in \mathbb{N}_0; \phi \in P; z \in \mathbb{U}).$$

Definition 2: Le $\phi(z) = (1+(1-2\alpha)z)/(1-z)$, then $K_\lambda^n(\phi) \equiv R_\lambda^n(\alpha)$ be the class of functions $f \in \mathcal{A}$ for which

$$\operatorname{Re}\left\{\frac{z(\mathfrak{D}_\lambda^n f(z))'}{\mathfrak{D}_\lambda^n f(z)}\right\} > \alpha, \tag{8}$$

$$(n, \lambda \in \mathbb{N}_0; 0 \le \alpha < 1; z \in \mathbb{U}).$$

Note that $K_0^0(\phi) \equiv S^*(\phi)$ were introduced and studied by Ma and Minda[5], $R_\lambda^0(\alpha) \equiv R_\lambda(\alpha)$ were studied by Ahuja[1] and $R_0^n(\alpha) \equiv R_n(\alpha)$ were studied by Kadioğlu[4]. Also for different choices of n, λ, and ϕ, we obtain several subclasses of analytic functions investigated earlier by other authors.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ consisting of the functions that can be expressed in the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k . \tag{9}$$

Finally, we defined the class $\mathcal{M}_\lambda^n(\alpha) = R_\lambda^n(\alpha) \cap \mathcal{T}$. Note that $\mathcal{M}_\lambda^n(\alpha) \subset R_\lambda^n(\alpha)$.

In this paper, we investigate several inclusion properties for the classes $K_\lambda^n(\phi(z))$ associated with the operator $\mathfrak{D}_\lambda^n$. Some applications involving operator are also obtained. Also, we derive several interesting properties of functions belonging to the $\mathcal{M}_\lambda^n(\alpha)$ consisting of analytic and univalent functions with negative coefficients. Coefficient inequalities, distortion theorems and result on integral operators are also given.

THE CLASSES $K_\lambda^n(\phi(z))$

To derive our first theorem, we need the following lemma due to Eenigenburg et al.[3].

Lemma 1: Let β, ν be complex numbers. Let $\phi \in P$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\operatorname{Re}[\beta\phi(z) + \nu] > 0$, $z \in \mathbb{U}$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta\phi(z)+\nu} \prec \phi(z) \Rightarrow p(z) \prec \phi(z),\ (z \in \mathbb{U}).$$

Theorem 1: Let $n, \lambda \in \mathbb{N}_0$ and $\phi \in P$. Then

$$K_{\lambda+1}^n(\phi) \subset K_\lambda^n(\phi).$$

Proof: Let $f \in K_{\lambda+1}^n(\phi)$ and set

$$p(z) = \frac{z(\mathfrak{D}_\lambda^n f(z))'}{\mathfrak{D}_\lambda^n f(z)} \tag{10}$$

where $p(z)$ analytic in $\mathbb{U}$ with $p(0) = 1$.

One can easily verify the identity

$$z(\mathfrak{D}_\lambda^n f(z))' = (\lambda+1)\mathfrak{D}_{\lambda+1}^n f(z) - \lambda \mathfrak{D}_\lambda^n f(z). \tag{11}$$

By using (11) in (10), we get

$$(\lambda+1)\frac{\mathfrak{D}_{\lambda+1}^{n}f(z)}{\mathfrak{D}_{\lambda}^{n}f(z)} = p(z)+\lambda . \qquad (12)$$

Taking the logarithmic differentiation on both sides of (12) and multiplying by z, we have

$$\frac{z(\mathfrak{D}_{\lambda+1}^{n}f(z))'}{\mathfrak{D}_{\lambda+1}^{n}f(z)} = p(z)+\frac{zp'(z)}{p(z)+\lambda} \quad (z\in\mathbb{U}). \quad (13)$$

Applying Lemma 1 to (13), it follows that $p\prec\phi$, that is $f\in K_{\lambda}^{n}(\phi)$. Therefore, we complete the proof of Theorem 1.

Corollary 1: Let $n,\lambda\in\mathbb{N}_0$ and $\phi\in P$. Then $K_{\lambda+1}^{n+1}(\phi)\subset K_{\lambda}^{n}(\phi)$.

Theorem 2: Let the function $f\in K_{\lambda}^{n}(\phi)$ and let c be real number such $c>-1$, then the function F defined by

$$F(z)=\frac{c+1}{z^{c}}\int_0^z t^{c-1}f(t)dt \qquad (14)$$

belongs to the class $K_{\lambda+1}^{n}(\phi)$.

Proof: Let $f\in K_{\lambda+1}^{n}(\phi)$. Then

$$\frac{z(\mathfrak{D}_{\lambda+1}^{n}F(z))'}{\mathfrak{D}_{\lambda+1}^{n}F(z)}\prec\phi(z). \qquad (15)$$

Set

$$p(z)=\frac{z(\mathfrak{D}_{\lambda}^{n}F(z))'}{\mathfrak{D}_{\lambda}^{n}F(z)}.$$

From the representation of $F(z)$, it follows that

$$z(\mathfrak{D}_{\lambda}^{n}F(z))'=(c+1)\mathfrak{D}_{\lambda}^{n}f(z)-c\mathfrak{D}_{\lambda}^{n}F(z).$$

By using the same technique as in the proof of Theorem 1, we get

$$\frac{z(\mathfrak{D}_{\lambda}^{n}f(z))'}{\mathfrak{D}_{\lambda}^{n}f(z)} = p(z)+\frac{zp'(z)}{p(z)+c} \quad . \qquad (16)$$

By applying Lemma 1 we obtain the required result.

THE CLASSES $\mathcal{M}_{\lambda}^{n}(\alpha)$

First, we provide a sufficient condition for a function f analytic in $\mathbb{U}$ to be in $\mathcal{M}_{\lambda}^{n}(\alpha)$.

Coefficient estimates:

Theorem 3: Let the function f be defined by (9). Then $f\in\mathcal{M}_{\lambda}^{n}(\alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)k^{n}C(\lambda,k)\,|a_k|\le 1-\alpha, \qquad (17)$$

where $n,\lambda\in\mathbb{N}_0$ and $C(\lambda,k)=\binom{k+\lambda-1}{\lambda}$.

Proof: Assume that the inequality (17) holds true and $|z|=1$. Then we obtain

$$\left|\frac{z(\mathfrak{D}_{\lambda}^{n}f(z))'}{\mathfrak{D}_{\lambda}^{n}f(z)}-1\right| = \left|\frac{\sum_{k=2}^{\infty}(k-1)k^{n}C(\lambda,k)a_k z^k}{z-\sum_{k=2}^{\infty}k^{n}C(\lambda,k)a_k z^k}\right|$$
$$\le\frac{\sum_{k=2}^{\infty}(k-1)k^{n}C(\lambda,k)\,|a_k|}{1-\sum_{k=2}^{\infty}k^{n}C(\lambda,k)\,|a_k|}$$
$$\le 1-\alpha.$$

This show that the values of $\frac{z(\mathfrak{D}_{\lambda}^{n}f(z))'}{\mathfrak{D}_{\lambda}^{n}f(z)}$ lies in a circle centered at $w=1$ whose radius 1 whose radius $1-\alpha$. Hence f satisfies the condition (17).

Conversely, we assume that the function f defined by (9) is in the class $\mathcal{M}_{\lambda}^{n}(\alpha)$. Then

$$\mathrm{Re}\left\{\frac{z(\mathfrak{D}_{\lambda}^{n}f(z))'}{\mathfrak{D}_{\lambda}^{n}f(z)}\right\}=\mathrm{Re}\left\{\frac{z-\sum_{k=2}^{\infty}k^{n+1}C(\lambda,k)a_k z^k}{z-\sum_{k=2}^{\infty}k^{n}C(\lambda,k)a_k z^k}\right\}>\alpha. \qquad (18)$$

For $z\in\mathbb{U}$, we choose values of z on the real axis so that $\frac{z(\mathfrak{D}_{\lambda}^{n}f(z))'}{\mathfrak{D}_{\lambda}^{n}f(z)}$ is real.

Upon clearing the denominator in (18) and letting $z\to 1^-$ through real values, we obtain

$$1-\sum_{k=2}^{\infty}k^{n+1}C(\lambda,k)\,|a_k|\ge\alpha\left\{1-\sum_{k=2}^{\infty}k^{n}C(\lambda,k)\,|a_k|\right\}$$

which gives (17).

Finally the result is sharp with the extremal function f given by

$$f(z)=z-\frac{1-\alpha}{(k-\alpha)k^{n}C(\lambda,k)}z^k, \qquad (20)$$
$$(n,\lambda\in\mathbb{N}_0;0\le\alpha<1;k\ge 2).$$

Corollary 2: Let the function f defined by (9) be in the class $\mathcal{M}_{\lambda}^{n}(\alpha)$. Then we have

$$a_k\le\frac{1-\alpha}{(k-\alpha)k^{n}C(\lambda,k)} \quad (n,\lambda\in\mathbb{N}_0;0\le\alpha<1;k\ge 2). \qquad (21)$$

This equality is attained for the function f given by (20).

Distortion theorem:

A distortion property for function f to be in the class $\mathcal{M}_\lambda^n(\alpha)$ given as follows:

Theorem 4: Let the function f defined by (9) be in the class $\mathcal{M}_\lambda^n(\alpha)$. Then for $|z|=r$ we have

$$r-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}r^2\le|f(z)|\le r+\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}r^2, \quad (22)$$

and

$$1-\frac{1-\alpha}{(2-\alpha)2^{n-1}(\lambda+1)}r\le|f'(z)|\le r+\frac{1-\alpha}{(2-\alpha)2^{n-1}(\lambda+1)}r.$$

Proof: In view of Theorem 4, we have

$$\sum_{k=2}^{\infty}a_k\le\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}.$$

Hence

$$|f(z)|\le r+\sum_{k=2}^{\infty}|a_k|r^k\le r+\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}r^2,$$

and

$$|f(z)|\ge r-\sum_{k=2}^{\infty}|a_k|r^k\ge r-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}r^2.$$

In the same way we have

$$1-\frac{1-\alpha}{(2-\alpha)2^{n-1}(\lambda+1)}r\le|f'(z)|\le r+\frac{1-\alpha}{(2-\alpha)2^{n-1}(\lambda+1)}r.$$

This completes the proof of the theorem. The above bounds are sharp. Equalities are attended for the following function

$$f(z)=z-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}z^2,\quad z=\pm r. \quad (23)$$

Corollary 3: The disk $|z|<1$ is mapped onto a domain that contains the disk

$$|w|<1-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}.$$

The result is sharp with extremal function (23).

Proof: The result follows upon letting $r\to 1$ in (22).

Integral Operator:

Bernardi[5] introduced integral operator defined as follows:

Let $f\in\mathcal{A}$ and $c>-1$. Then, for $z\in\mathbb{U}$

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt$$

Now we consider our results.

Theorem 5: Let the function f defined by (9) be in the class $\mathcal{M}_\lambda^n(\alpha)$ and let c be real number such that $c>-1$, then the function F defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (24)$$

Proof: From the representation of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}|b_k|z^k$$

where $|b_k|=\left(\frac{c+1}{c+k}\right)|a_k|<1$. Therefore

$$\sum_{k=2}^{\infty}(k-\alpha)k^nC(\lambda,k)|b_k|$$
$$=\sum_{k=2}^{\infty}(k-\alpha)k^nC(\lambda,k)\left(\frac{c+1}{c+k}\right)|a_k|$$
$$\le\sum_{k=2}^{\infty}(k-\alpha)k^nC(\lambda,k)|a_k|\le 1-\alpha.$$

Since $f\in\mathcal{M}_\lambda^n(\alpha)$ and hence by Theorem 5, $F\in\mathcal{M}_\lambda^n(\alpha)$.

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