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# Proof of Bernhard Riemann's Functional Equation using Gamma Function

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**Abstract:** This study shows the use of gamma function to prove the Riemann functional equation. Two approaches had been used to solve this problem: first the value of t in the definition of the gamma function had been changed to pi nu x if only if sigma is greater than zero in the complex plane. Secondly, the Poisson summation formula is used to show that zeta has a simple pole at s = 1 with residue 1, we had found that Riemann zeta function depended intimately on properties of gamma function, which was a new gate for solving complex problems related to zeta function.

**Key words:** Gamma function, specific vertical line

## INTRODUCTION

Bernhard Riemann's paper, Ueber die Anzahl der primzahlen unter einer gegebenen Grösse (On the number of primes less than a given quantity) was first published in Monatsberichte de Berliner AKademie, in November 1859. This study, just six manuscript pages in length, introduced radically new idea to the study of prime numbers, ideas which led, in 1896, to independent proofs by Hadamard and de la Valleé Poussin of the prime number theorem. This theorem, first conjectured by Gauss when he was a young man, states that the number of primes less than x is asymptotic to x/log(x). Very roughly speaking, this means that the probability that a randomly chosen number of magnitude x is a prime is 1/log(x). Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x and the roots (zeros) of the zeta function, defined by:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} \quad (1)$$

He also formulated a conjecture about the location of these zeros, which fall into two classes: the "obvious zeros" -2, -4, -6, -8 and those real part lies between 0 and 1. He checked the first few zeros zeta function by hand and they satisfy his hypothesis. By now over 2.5 billion zeros have been checked by computer. Very strong experimental evidence, but in mathematical we require a proof. A proof gives certainty, but, just as important, it helps us to understand why a result is true. It is in this direction that we used the gamma function to proof Riemann functional equation, where the gamma can be thought of as the natural way to generalize the concept of factorial to non-integer arguments such as:

- For non-negative integer n, denote n! can be defined by:

$$n! = \prod_{n=1}^{n} r$$

where, for n = 0 the empty product is taken to be 1

- For every non-integer n we have:

$$\Gamma(n+1) = n!$$

where, $\Gamma$ is Euler gamma function who come up with a formula for such a generalization in 1729. At around the same time, James Stirling independently arrived at a different formal but was unable to show that it always converged. The process of our proof is compared to other studies on the open literature on the functional equation.

## MATERIALS AND METHODS

**Riemann functional equation:** The function $\zeta(s)$ can be continued analytically over the whole complex plane C and satisfies the functional equation:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-s) \quad (2)$$

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where, $\Gamma$ denotes the gamma function. In particular, the function $\zeta(s)$ is analytic everywhere, for a single pole at s = 1 with residue 1.

**Proof:** First note that the functional Eq. 2 enables properties of $\zeta(s)$ for $\sigma < 0$ to be inferred from properties of $\zeta(s)$ for $\sigma > 1$ as can be observed from (2) the study of Riemann zeta function depends on properties of the gamma function:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \qquad (3)$$

In Riemann published paper, he formulated a conjecture about the location of zeros of the zeta function, which fall into two classes: the "obvious zeros" -2,-4,-6,-8 and those whose real part lies between 0 and 1.

He added also in a conjecture that the real part of the non obvious zero is exactly 1/2. That is, they all lie on a specific vertical line in the complex plane Fig. 1.

If the Riemann conjecture is true, the value of s also lie on that specific vertical line in the complex plane, and we need to find the specific vertical line that Riemann referred to in his paper as our study is going to based on it. From Fig. 1 we found that:

$$\text{specific vertical line} = \frac{\text{vertical axis}}{2}$$

That is, from the origin to positive $y$ axis, thus the value of s can be replaced by S/2.

Now the specific vertical line in the complex plane is known, we can suppose that $\sigma > 0$.

Writing $t = \pi n^2 x$ with $dt = \pi n^2 dx$

then substituting t and dt in the definition of the gamma function $\Gamma$ Eq. 3, we have:

$$\Gamma(\frac{S}{2}) = \int_0^\infty e^{-\pi n^2 x} (\pi n^2 x)^{\frac{s}{2}-1} \pi n^2 dx$$

$$= \int_0^\infty \pi n^2 (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$= \int_0^\infty (\pi \cdot \pi^{(\frac{s}{2}-1)} n^2 \cdot n^{2(\frac{s}{2}-1)} x^{\frac{s}{2}-1}) e^{-\pi n^2 x} dx$$

$$= \pi^{\frac{s}{2}} n^s \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx \qquad (4)$$

$$\frac{\Gamma(\frac{S}{2})}{\pi^{\frac{s}{2}} n^s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}} \Gamma(\frac{S}{2}) n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

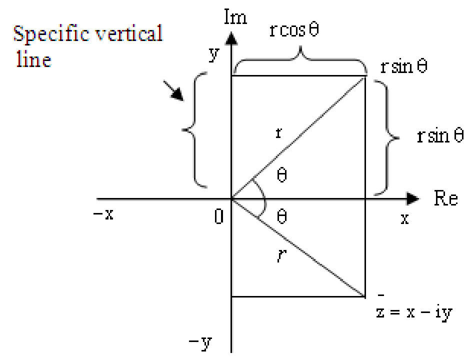


Fig. 1: The geometric representation of z and its conjugate z in the complex plane. The distance along the black line from origin to the point z is the modulus or absolute value of z. The angle θ is the argument of z

In the distribution of prime numbers, Riemann extended Euler's zeta function to the entire complex plane that:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty n^{-s} \qquad (5)$$

Now by multiplying both the left hand side and right hand side of Eq. 4 by:

$$\sum_{n=1}^\infty$$

we have:

$$\pi^{-\frac{s}{2}} \Gamma(\frac{S}{2}) \sum_{n=1}^\infty n^{-s} = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}} \Gamma(\frac{S}{2}) \sum_{n=1}^\infty n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} (\sum_{n=1}^\infty e^{-\pi n^2 x}) dx \qquad (6)$$

$$\pi^{-\frac{s}{2}} \Gamma(\frac{S}{2}) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} (\sum_{n=1}^\infty e^{-\pi n^2 x}) dx$$

where, the change of order of summation and integration is justified by the convergence of:

$$\sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

Writing

$$\omega(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} dx \qquad (7)$$

it follows for $\sigma > 1$ Eq. 6 becomes:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}\omega(x)dx + \int_{\beta}^{\alpha} y^{\frac{s}{2}-1}\omega(y)dy \quad (8)$$

Due to the location of the non-obvious zeros values that lie on the specific vertical line in the complex plane according to Riemann Fig1, we can write that:

$$y = \frac{1}{x} = x^{-1} \quad \text{and} \quad dy = -\frac{dx}{x^2}$$

Substituting the value of $y$ and $dy$ in the second term of the integral Eq. 8 from $\beta$ to $\alpha$ we have:

$$-\int_{\beta}^{\alpha} (x^{-1})^{\frac{s}{2}-1}\omega(x^{-1})x^{-2}dx$$

Using the integration properties by inversing the integration borders from $\alpha$ to $\beta$ instead from $\beta$ to $\alpha$. We have:

$$\begin{aligned}&\int_{\alpha}^{\beta} (x^{-1})^{\frac{s}{2}-1}\omega(x^{-1})x^{-2}dx\\ &= \int_{\alpha}^{\beta} x^{-1(\frac{s}{2}-1)}\cdot x^{-2}\omega(x^{-1})dx\\ &= \int_{\alpha}^{\beta} x^{-\frac{s}{2}-1}\omega(x^{-1})dx\end{aligned} \quad (9)$$

Substituting Eq.9 into the second term of the right hand side of Eq.8 and letting $\alpha = 1$ and $\beta = \infty$, we have:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s) = \int_1^{\infty} x^{\frac{s}{2}-1}\omega(x)dx + \int_1^{\infty} x^{-\frac{s}{2}-1}\omega(x^{-1})dx \quad (10)$$

Now we have to show that for every x > 0, the function:

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 1 + 2\omega(x) \quad (11)$$

Satisfies the functional equation $\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x)$ which can be written:

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi\frac{1}{x}} = x^{\frac{1}{2}}\sum e^{-n^2\pi x} \quad (12)$$

Hence:

$$\begin{aligned}2\omega(x^{-1}) &= \theta(x^{-1}) - 1\\ &= x^{\frac{1}{2}}\theta(x^{-1}) - 1\\ &= -1 + x^{\frac{1}{2}} + 2x^{\frac{1}{2}}\omega(x)\\ \omega(x^{-1}) &= -\frac{1}{2} + \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)\end{aligned} \quad (13)$$

substituting $\omega(x^{-1})$ into Eq.9 where $\alpha = 1$ and $\beta = \infty$ we have:

$$\int_1^{\infty} x^{-\frac{s}{2}-1}\omega(x^{-1})\,dx$$

We obtain:

$$\begin{aligned}&\int_1^{\infty} x^{-\frac{s}{2}-1}\omega(x^{-1})\,dx\\ &= \int_1^{\infty} x^{-\frac{s}{2}-1}(-\frac{1}{2} + \frac{x^{\frac{1}{2}}}{2} + x^{\frac{1}{2}}\omega(x))\,dx\\ &= -\frac{1}{s} + \frac{1}{s-1} + \int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}}\omega(x)\,dx\end{aligned} \quad (14)$$

It follows on combining Eq. 9 and Eq. 14 that for $\sigma > 1$ Eq. 8 becomes:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})\omega(x)dx \quad (15)$$

The integral on the right hand side of Eq. 15 converges absolutely for any s, and uniformly in any bounded part of the plane, since $\omega(x) = O(e^{-\pi x})$ when $x \to +\infty$. Hence the integral represents an entire function of s, and the formula gives the analytic continuation of $\zeta(s)$ over the whole plane. Note that the right hand side of Eq. 14 remains unchanged when s, is replaced by 1-s, so that the functional Eq. 2 follows immediately. Finally note that the function:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s) \quad (16)$$

is analytic everywhere, since $s\Gamma(s)$ has non zeros, the only possible pole of $\zeta(s)$ is at s = 1 with residue 1 as we have shown in the previous equations. It remains to establish the functional equation Eq. 12 for every $x > 0$.

Which the starting point is the Poisson summation formula, that under certain conditions on a function f (t) we have:

$$\sum_{A\le n\le B}{}' f(n) = \sum_{v=-\infty}^{\infty}\int_A^B f(t)e^{2\pi i v t}dt \tag{17}$$

Where, $\sum_{A\le n\le B}{}'$ denotes that the terms in the sum corresponding to $n = A$ and $n = B$ are $\frac{1}{2}f(A)$ $\frac{1}{2}f(B)$ and respectively. Using (17) with N = -A and :

N = B and $f(t) = e^{-t^2\frac{\pi}{x}}$

We have:

$$\sum_{n=-N}^{N} e^{-n^2\frac{\pi}{x}} = \sum_{v=-\infty}^{\infty}\int_{-N}^{N} e^{-t^2\frac{\pi}{x}}e^{2\pi i v t}dt \tag{18}$$

Letting $N \to \infty$, we obtain:

$$\sum_{n=-\infty}^{\infty} e^{-n^2\frac{\pi}{x}} = \sum_{v=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t^2\frac{\pi}{x}}e^{2\pi i v t}dt \tag{19}$$

This is justified by the fact that:

$$\begin{aligned}&\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)e^{-t^2\frac{\pi}{x}}e^{2\pi i v t}dt\\ &\qquad = 2\int_N^{\infty} e^{-t^2\frac{\pi}{x}}\cos(2\pi v t)dt,\end{aligned} \tag{20}$$

and that:

$$\left|\sum_{v\ne 0}\int_N^{\infty} e^{-t^2\frac{\pi}{x}}\cos(2\pi v)\,dt\right| \to 0 \text{ as } N \to \infty.$$

now writing $t = ux$ and using (19), we have:

$$\sum_{n\sim-\infty}^{\infty} e^{-n^2\frac{\pi}{x}} = x\sum_{v=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2\pi}e^{2\pi i v x u}du$$

$$\begin{aligned}&= x\sum_{v=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u-iv)^2\pi x - v^2\pi x}du\\ &= x\sum_{v=-\infty}^{\infty} e^{-v^2\pi x}\int_{-\infty}^{\infty} e^{-(u-iv)^2\pi x}du.\end{aligned} \tag{21}$$

The function $e^{-z^2\pi x}$ is an entire function of the complex variable z and from Cauchy's theorem, we have:

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-(u-iv)^2\pi x}du\\ &= \int_{-\infty}^{\infty} e^{u^2\pi x}du = Ax^{\frac{1}{2}}\end{aligned} \tag{22}$$

Where:

$$A = \int_{-\infty}^{\infty} e^{-y^2\pi}dy = 1 \tag{23}$$

The functional Eq. 12 now follows on combining Eq. 21-23 and the proof of Eq. 2 is completed.

## RESULTS AND DISCUSSION

For more than two thousand years, mathematics has been a part of the human search for understanding. Mathematical discoveries have come both from the attempt to describe the natural world and from the desire to arrive at a form of inescapable truth from careful reasoning. These remain fruitful and important motivations for mathematical thinking, but in the last century mathematics has been successfully applied to many other aspects of the human world. Such as, voting trends in politics, the dating of ancient artifacts, the analysis of automobile traffic patterns, and long-term strategies for the sustainable harvest of deciduous forests, to mention a few. Today, mathematics as a mode of thought and expression is more valuable than ever before. Due to the importance or involvement of Mathematics in other scientific domains, many papers in the open literature have tried to prove the Riemann functional equation whose most of these papers have encountered difficulties to give a real proof.

For example, in page 2 of[1] the author did a great effort to prove the Riemann functional equation, but unfortunately it was unclear and unfinished. When arrived at below step:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1}\omega(x)dx \tag{24}$$

The author wrote "we take the natural step of splitting the integration at $x = 1$ then substituting 1/x for x in the integral from 0-1 ", where the integral from

0-1 has not been indicated and no reference has been also made on the value of dx(dx = $-x^{-2}$) and furthermore no explanation about how the left hand side of Eq. 24 has been obtained. Based on the location of non-obvious zeros on the complex plane the borders of Eq. 24 should be from 1-∞ instead of 0-∞. In our proof, we have focused on the clue given by Riemann in his conjecture states that, the zeroes of the Riemann zeta function that are inside the Critical Strip. That is, the vertical strip of the complex plane where the real part of the complex variable is in [0; 1], are actually located on the Critical line. That is the specific vertical line of the complex plane with real part equal to½.

## CONCLUSION

In this study, the gamma function is used to prove the Riemann functional equation, based on the real part of the non-obvious zeros that is exactly 1/2; following by the use of Poisson summation formula to show that zeta has a simple pole at s = 1 with residue 1. From this research we have found that the riemann zeta function depends on properties of the gamma function when sigma is greater than zero in the complex plane and that the non-obvious zeros all lie on the specific vertical line in the complex plane. This new founding may help to solve some complex problems related to zeta function.

## REFERENCES


1. Kedlaya, K.S., 2007. The functional equation for riemann zeta function. J. Anal. Theor., 1-5. http://www-math.mit.edu/~kedlaya/18.785/funceq.pdf
2. Bochner, S., 1958. On riemann's functional equation with multiple gamma factors. Ann. Math., 67: 29-41.
3. Sondow, J., 1994. Analytic continuation of riemann's zeta function and values at negative integers via euler's transformation of series. Proc. Am. Math. Soc., 12: 421-424.
4. Spanier, J. and K.B. Aldham, 1987. The Zeta Numbers and Related Function: An Atlas of Function. 1st Edn., Taylor and Francis, Hemispher, Washington, DC., ISBN: 978-0891165736, pp: 700.
5. Srivastava, H.M., 2000. Some simple algorithms for the evaluations and representations of the zeta function at positive integer arguments. J. Math. Anal. Appl., 246: 331-351. http://library.wolfram.com/infocenter/Articles/2024/
6. Caldwell, C. and H. Dubner, 1998. Primes in pi. J. Recreat. Math., 29: 282-289. http://www.utm.edu/staff/caldwell/preprints/PrimesInPi.pdf
7. Brian Conrey, J., 2003. The riemann hypothesis. Notic. Am. Math. Soc., 50: 341-353. http://direct.bl.uk/bld/PlaceOrder.do?UIN=127249570&ETOC=RN&from=searchengine
8. Molhem, H. and R. Pourgoli, 2008. A numerical algorithm for solving a one-dimensional inverse heat conduction problem. Am. J. Math. Stat., 4: 98-101. http://www.scipub.org/fulltext/jms2/jms24298-101.pdf
9. Ayoub, R., 1974. Euler and the zeta function. Am. Math. Monthly, 81: 1067-1086.
10. Biane, P., J. Pitman and M. Yor, 2001. Probability laws related to the jacobi theta and riemann zeta functions and brownian excursions. Bull. Am. Math. Soc., 38: 435-465. http://www.ams.org/bull/2001-38-04/S0273-0979-01-00912-0/home.html
11. Bloch, S., 1996. Zeta values and differential operators on the circle. J. Algebra, 182: 476-500. DOI: 10.1006/jabr.1996.0182