Journal of Mathematics and Statistics: 4(3): 156-160, 2008
ISSN: 1549-3644

On Cesáro means for Fox-Wright functions

Maslina Darus and Rabha W. Ibrahim
School of Mathematical Sciences, Faculty of Science and Technology
University Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

Abstract: The polynomial approximants which retain the zero free property of a given analytic function involving fox-wright function in the unit disk U: = {z: |z|<1} is found. The convolution methods of a geometric function that the Cesáro means of order μ retains the zero free property of the derivatives of bounded convex functions in the unit disk are used. Other properties are also established.

Key words: Fox-wright function, cesáro sum, convolution

INTRODUCTION

In the theory of approximation, the important problem is to find a suitable finite (polynomial) approximation for the outer infinite series f so that the approximant reduces the zero-free property of f Recall that an outer function (zero-free) is a function fεH of the form:

$$f(z) = e^{i\gamma} e^{1/2\pi} f_{-\pi}^{\pi} \frac{1+e^{it}z}{1-e^{it}z} \log \psi(t) dt$$

where, $\psi(t) \geq 0, \log\psi(t)$ is in L^1 and $\Psi(t)$ is in L^{p}[3]. Outer function plays an important role in H^p theory, arises in characteristic equation which determines the stability of certain nonlinear systems of differential equations[2]. We observed that for outer functions, the standard Taylor approximants do not, in general, retain the zero-free property of f It was shown in[1] that the Taylor approximating polynomials to outer functions can vanish in the unit disk. By using convolution methods, the classical Cesáro means retain the zero-free property of the derivatives of bounded convex functions in the unit disk. The classical Cesáro means play an important role in geometric function theory[5-7]. In this study, we obtain new Cesáro approximants for outer functions. Indeed, fox-wright function is involved and stated as follows:

For complex parameters:

$$\alpha 1, \ldots, \alpha_q (\frac{\alpha_j}{A_j} \neq 0, -1, -2, \ldots; j = 1, \ldots, q)$$

And

$$\beta_1, \ldots, \beta_p (\frac{\beta_j}{B_j} \neq 0, -1, -2, \ldots; j = 1, \ldots, p),$$

the fox-wright generalization ${}_q\psi_p[z]$ of the hypergeometric ${}_qF_q$ function by[4,10,11]:

$$
\begin{aligned}
{}_q\psi_p\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_q, A_q); \\ (\beta_1, B_1), \ldots, (\beta_p, B_p); z\end{matrix}\right] &= {}_q\psi_p\left[(\alpha_j, A_j)_{1,q}; (\beta_j B_j)_{1,p}; z\right] \\
&:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1)\ldots\Gamma(\alpha_q + nA_q) z^n}{\Gamma(\beta_1 + nB_1)\ldots\Gamma(\beta_p + nB_p) n!} \\
&= \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{q} \Gamma(\alpha_j + nA_j) z^n}{\Pi_{j=1}^{p} \Gamma(\beta_j + nB_j) n!}
\end{aligned}
$$

where, $A_j>0$ for all $j = 1, \ldots, q, B > 0$ for all j = 1,..,p and nship $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$ for suitable values $|z|$ For special case, when $A_j = 1$ for all j = 1 and B_j=1 for all j = 1,…,p we have the following relationship:

$${}_qF_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_p; z) = \Omega\, {}_q\Psi_p[(\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,p}; z]$$

$$q \leq p + 1; q, p \in N_0 = N \cup \{0\}, z \in U$$

Where:

$$\Omega := \frac{\Gamma(\beta_1)\ldots\Gamma(\beta_p)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_q)}$$

Let A be the class of Fox-Wright functions in the unit disk $U := \{z : |z| < 1\}$ take the form:

Corresponding Author: Maslina Darus, School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

$$\varphi(z) := z_q\Psi_p[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,p}; z] = \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)z^{n+1}}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)n!}, z \in U \quad (1)$$

With:

$$0 < \prod_{j=1}^{q}\Gamma(\alpha_j + nA_j) \le \prod_{j=1}^{p}\Gamma(\beta_j + nB_j) \quad (2)$$

RESULTS AND DISCUSSION

This class of function is a generalization to the one studied by[5]. The author observed the following results:

Lemma 1: Let $0 < \alpha \le \beta$ If $\beta \ge 2$ or $\alpha + \beta \ge 3$ then the function of the form $f(z) = \sum_{n=0}^{\infty} \frac{(\alpha)n}{(\beta)n} z^{n+1}, z \in U$ is convex.

Note that (x) is the Pochhammer symbol defined by:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{Bmatrix} 1, n = 0 \\ x(x+1)...(x+n-1), n = \{1,2,...\} \end{Bmatrix}$$

Lemma 2: Assume that $a_1 = 1$ and $a_n \ge 0$ for $n \ge 0$ such that $\{a_n\}$ is a convex decreasing sequence i.e.:

$$a_n - 2a_{n+1} + a_{n+2} \ge 0 \text{ and } a_{n+1} - a_{n+2} \ge 0$$

Then

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2}, z \in U$$

We apply Lemma 1.2, to find the next result which is a generalization to[Lemma 5; 8].

Lemma 3: Let (2) holds. Then:

$$\Re\left\{\frac{\varphi(z)}{z}\right\} > \frac{1}{2} \text{for all } z \in U$$

Proof: From the definition of the function $\varphi(z)$ we have:

$$\frac{\varphi(z)}{z} = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_q)}{\Gamma(\beta_1)...\Gamma(\beta_p)} + \sum_{n=2}^{\infty} B_n z^{n-1}$$

Where:

$$B_n := \frac{\Gamma(\alpha_1 + (n-1)A_1)...\Gamma(\alpha_q + (n-1)A_q)1}{\Gamma(\beta_1 + (n-1)B_1)...\Gamma(\beta_p + (n-1)B_p)\Gamma(n)}$$

for $n \ge 2$. From (2), we have $B_n > 0$ for all $n \in N$ Now by using Gauss's multiplication theorem for the Gamma function, it follows that:

$$\Gamma[k(n+\gamma)] = \Gamma(k\gamma)(k^k)^n(\gamma)_n(\gamma + \frac{1}{k})_{n...}(\gamma + \frac{k-1}{k})_n$$

for positive integer k and non-negative integer n Thus since we obtain[9]:

$$\Gamma(\alpha + A_n) = \Gamma(\alpha)(\frac{\alpha}{A})_n(\frac{\alpha+1}{A})_n...(\frac{\alpha+A-1}{A})_n(A^A)^n$$

Also, by using the fact that $(x)_n = (x)_{n-1}(x+n-1)$, we find:

$$B_n = \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j)(\frac{\alpha_j}{A_j})_{n-1}(\frac{\alpha_j+1}{A_j})_{n-1}...(\frac{\alpha_j+A_j-1}{A_j})_n(A_j^{A_j})^{n-1}}{\Pi_{j=1}^{p}\Gamma(\beta_j)(\frac{\beta_j}{B_j})_{n-1}(\frac{\beta_j+1}{B_j})_{n-1}...(\frac{\beta_j+B_j-1}{B_j})_{n-1}(B_j^{Bj})^{n-1}} \quad (3)$$

Then we obtain $B_n \ge 0$. Moreover, we have B_{n+1} and B_{n+2} in terms of B_n:

$$B_{n+1} = \frac{\Gamma(\alpha_1 + nA_1)...\Gamma(\alpha_q + nA_q)1}{\Gamma(\beta_1 + nB_1)...\Gamma(\beta_p + nB_p)n\Gamma(n)}$$

$$= \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j)(\frac{\alpha_j}{A_j})n(\frac{\alpha_j+1}{A_j})n...(\frac{\alpha_j+A_j-1}{A_j})n(A_j^{Aj})^n}{\Pi_{j=1}^{p}\Gamma(\beta_j)(\frac{\beta_j}{B_j})n(\frac{\beta_j+1}{B_j})n...(\frac{\beta_j+B_j-1}{B_j})n(B_j^{Bj})^n}\frac{1}{n\Gamma(n)} \quad (4)$$

$$= \frac{\Pi_{j=1}^{q}(\frac{\alpha_j}{A_j}+n-1)(\frac{\alpha_j+1}{A_j}+n-1)...(\frac{\alpha_j+A_j-1}{A_j}+n-1)(A_j^{A_j})}{\Pi_{j=1}^{p}(\frac{\beta_j}{B_j}+n-1)(\frac{\beta_j+1}{B_j}+n-1)...(\frac{\beta_j+B_j-1}{B_j}+n-1)(B_j^{B_j})}$$

$$\frac{B_n}{n\Gamma(n)}$$

and

$$B_{n+2} = \frac{\Gamma(\alpha_1 + (n+1)A_1)...\Gamma(\alpha_q + (n+1)A_q)}{\Gamma(\beta_1 + (n+1)B_1)...\Gamma(\beta_p + (n+1)B_p)}\frac{1}{n(n+1)\Gamma(n)}$$

$$= \frac{\Pi_{j=1}^{q}(\frac{\alpha_j}{A_j}+n-1)(\frac{\alpha_j}{A_j}+n)(\frac{\alpha_j+1}{A_j}+n-1)(\frac{\alpha_j+1}{A_j}+n)}{\Pi_{j=1}^{p}(\frac{\beta_j}{B_j}+n-1)(\frac{\beta_j}{B_j}+n)(\frac{\beta_j+1}{B_j}+n-1)(\frac{\beta_j+1}{B_j}+n)}... \quad (5)$$

$$\frac{(\frac{\alpha_j+A_j-1}{A_j}+n-1)(\frac{\alpha_j+A_j-1}{A_j}+n)(A_j^{A_j})}{(\frac{\beta_j+B_j-1}{B_j}+n-1)(\frac{\beta_j+B_j-1}{B_j}+n)(B_j^{B_j})}\frac{B_n}{n(n+1)\Gamma(n)}$$

Thus from the assumption it follows that:

$$B_{n+1} - B_{n+2} = B_{n+1} \left(1 - \frac{\Pi_{j=1}^{q}\left(\frac{\alpha_j}{A_j}+n\right)\left(\frac{\alpha_j+1}{A_j}+n\right)\left(\frac{\alpha_j+A_j-1}{A_j}+n\right)\left(A_j^{Aj}\right)B_n}{\Pi_{j=1}^{p}\left(\frac{\beta_j}{B_j}+n\right)\left(\frac{\beta_j+1}{B_j}+n\right)\left(\frac{\beta_j+B_j-1}{B_j}+n\right)\left(B_j^{Bj}\right)n\left(n+1\right)\Gamma\left(n\right)} \right) \geq 0, \forall n \in \mathbb{N}$$

In the same way and by using (3) and (4), we can show that:

$$B_n - 2B_{n+1} + B_{n+2} \geq 0, \forall n \in \mathbb{N} \tag{6}$$

Thus we find the sequence $\{B_n\}$ is convex decreasing and in virtue of lemma 2, we obtain that:

$$\Re\left\{ \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_q)}{\Gamma(\beta_1)\ldots\Gamma(\beta_p)} + \sum_{n=2}^{\infty} B_n z^{n-1} \right\} = \Re\left\{ \frac{\varphi(z)}{z} \right\} > \frac{1}{2}$$

The proof is complete:

We define S*, C, QS* and QC the subclasses of A consisting of functions which are, respectively, starlike in U, convex in U, close-to-convex and quasi-convex in U. Thus by definition, we have:

$$S^* := \left\{ \varphi \in A : \Re\left(\frac{(z\varphi'(z))}{\varphi(z)} \right) > 0, z \in U \right\},$$

$$C := \left\{ \varphi \in A : \Re\left(1 + \frac{(z\varphi''(z))}{\varphi'(z)} \right) > 0, z \in U \right\},$$

$$QS^* := \left\{ \varphi \in A : \exists g \in S^* \text{ s.t. } \Re\left(\frac{(z\varphi'(z))}{g(z)} \right) > 0, z \in U \right\}$$

And

$$QC := \left\{ \varphi \in A : \exists g \in C \text{ s.t. } \Re\left(\frac{(z\varphi'(z))'}{g'(z)} \right) > 0, z \in U \right\}$$

It is easily observed from the above definitions that:

$$\varphi(z) \in C \Leftrightarrow z\varphi'(z) \in S^* \tag{7}$$

And

$$\varphi(z) \in QC \Leftrightarrow z\varphi'(z) \in QS^* \tag{8}$$

Note that $\varphi \in QS^*$ if and only if there exists a function $g \in S^*$ such that:

$$z\varphi'(z) = g(z)p(z) \tag{9}$$

where, $p(z) \in P$ the class of all analytic functions of the form:

$$p(z) = 1 + P_1 + z + p_2 z^2 + \ldots, \text{s.t.} p(0) = 1$$

Given two functions:

$$f, g \in A, f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution or Hadamard product f(z)*g(z) is defined by:

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U$$

We can verify the following result for $f \in A$ and takes the form (1).

Lemma 4[5]:

- If $\varphi \in C$ and $g \in S^*$ then $\varphi * g \in S^*$
- If $\varphi \in C$ and $g \in S^*, p \in P$ with p(0) = 1 then $\varphi * gp = (\varphi * g)p_1$ where $p_1(U) \subset$ close convex hull of p(U)

CONCLUSION

Cesáro approximants for outer functions: The Cesáro sums of order μ where $\mu \in \mathbb{N} \cup \{0\}$ of series of the form (1) can defined as:

$$\sigma_k^{\mu}(z,\varphi) = \sigma_k^{\mu} * \varphi(z) = \sum_{n=0}^{\infty} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j) z^{n+1}}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j) n!}$$

where, $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$.

We begin with the following result:

Theorem 1: Let $\varphi \in A$ be convex in U Then the Cesáro means $\sigma_k^{\mu}(z,\varphi), z \in U$ of order $\mu \geq 1$, of $\varphi'(z)$ are zero-free on U for all k.

Proof: In view of Lemma 1, the analytic function φ of the form (1) is convex in Uif

$$\prod_{j=1}^{p}\Gamma(\beta_j + nB_j) \ge 2 \text{or} \prod_{j=1}^{q}\Gamma(\alpha_j + nA_j) + \prod_{j=1}^{p}\Gamma(\beta_j + nB_j) \ge 3 \quad (10)$$

where, (2) holds. Let $\varphi(z) := \sum_{n=0}^{\infty}(n+1)z^{n+1}$ be defined such that:

$$z\varphi'(z) = \phi(z) * \varphi(z) = \sum_{n=0}^{\infty}\frac{(n+1)}{n!}\frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)}z^{n+1}$$

Then:

$$\begin{aligned}\sigma_k^{\alpha}(z,\varphi') &= \varphi'(z) * \sigma_k^{\mu}(z)\\ &= \frac{z\varphi'(z) * z\sigma_k^{\mu}}{z}\\ &= \frac{\varphi(z) * \varphi z * z\sigma_k^{\mu}}{z}\\ &= \frac{\varphi(z) * z(z\sigma_k^{\mu})'}{z}\end{aligned}$$

In view of Lemma 3, the relation (8) and the fact that $z\sigma_k^{\mu}$ is convex yield that there exists a function $g \in S^*$ and $p \in P$ with p(0) = 1 such that:

$$\frac{\varphi(z) * z(z\sigma_k^{\mu})'}{z} = \frac{\varphi(z) * gp(z)}{z} = \frac{(\varphi(z) * g(z))p_1(z)}{z} \ne 0$$

We know that $\Re\{p_1(z)\} > 0$ and that $\varphi(z) * g(z) = 0$ if and only if z = 0 Hence, $\sigma_k^{\mu}(z\varphi') \ne 0$ and the proof is complete.

Corollary 1: If f(U) is bounded convex domain, then the Cesáro means $\sigma_k^{\mu}(z), z \in U$ for the outer function $\varphi'(z)$ are zero-free onU for all k.

Proof: It comes from the fact that the derivatives of bounded convex functions are outer function [3]. The next result shows the upper and lower bound for $\sigma_k^{\mu}(z,\varphi')$.

Theorem 2: Let $\varphi \in A$ Assume that (2) and (10) hold. Then:

$$\frac{1}{2}|z| < |\sigma_k^{\mu}(z,\varphi')| \le \frac{(k+1)}{(k-1)!'}1 \le k < \infty, z \in U, z \ne 0$$

Proof: Under the conditions of the theorem, we have that f is convex (Lemma 1.1), then in virtue of Theorem 1, we obtain that $\sigma_k^{\mu}(z,\varphi') \ne 0$ thus $\sigma_k^{\mu}(z,\varphi') > 0$ Now by applying Lemma 1.3, on $\sigma_k^{\mu}(z,\varphi')$ and using the fact that $\Re\{z\} \le |z|$ and since:

$$\frac{\binom{k-n+\mu}{k-n}}{\binom{k+\mu}{k}} = \frac{k!(k-n+\mu)!}{(k-n)!(k+\mu)!} \le 1 \quad (11)$$

for $\mu \ge 0$ and n = 0, 1,…, k yield:

$$\frac{1}{2} < \Re\left\{\frac{\sigma_k^{\mu}(z,\varphi')}{z}\right\} \le \frac{|\sigma_k^{\mu}(z,\varphi')|}{z}, |z| > 0 \text{ and } z \in U$$

For the other side, we pose that:

$$\begin{aligned}|\sigma_k^{\mu}(z,\varphi')| &= |\varphi'(z) * \sigma_k^{\mu}(z)|\\ &= \left|\sum_{n=0}^{k}\frac{(n+1)}{n!}\frac{\binom{k-n+\mu}{k-n}}{\binom{k+\mu}{k}}\frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)}z^n\right|\\ &\le \sum_{n=0}^{k}\frac{(n+1)}{n!}\frac{\binom{k-n+\mu}{k-n}}{\binom{k+\mu}{k}}\frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)}|z^n|\\ &\le \sum_{n=0}^{k}\frac{(n+1)}{n!}\frac{\binom{k-n+\mu}{k-n}}{\binom{k+\mu}{k}}\frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)}\\ &\le \sum_{n=0}^{k}\frac{(n+1)}{n!} \le \frac{(k+1)}{(k-1)!'}k < \infty\end{aligned}$$

When, $n \to k$ Hence the proof. Finally, we give the following result:

Theorem 3: Let $\varphi \in A$ and let (2) holds. Then:

$$\lim_{k\to\infty}\sigma_k^{\alpha}(z,\varphi) = \frac{z}{(1-z)\lambda'}\lambda > 1, z \in U$$

Proof: From the assumption (2) and by (11) yield:

$$\left|\sigma_k^{\alpha}(z,\varphi) - \frac{z}{(1-z)\lambda}\right| = \left| \sum_{n=0}^{k} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)} \frac{1}{n!} z^{n+1} - \sum_{n=0}^{\infty} \frac{(\lambda)n}{n!} z^{n+1} \right|$$

$$= \frac{1}{n!} \left| \sum_{n=0}^{k} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)} - (\lambda)_n \right| \left| z^{n+1} - \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{n+1} \right|$$

$$\leq \left| \sum_{n=k+1}^{\infty} \frac{(\lambda)_n}{n!} - \sum_{n=1}^{k} \frac{(\lambda)_n}{n!} \right|$$

$$= 0 \text{ as } k \to \infty$$

ACKNOWLEDGEMENT

The work here was fully supported by eScience Fund: 04-01-02-SF0425, MOSTI, Malaysia.

REFERENCES

1. Barnard, R.W., J. Cima and K. Pearce, 1998. Cesáro sum approximation of outer functions. Ann. Uni. Marie Curie-Sklodowska Sect. A, 52: 1-7.
2. Cunningham, W., 1958. Introduction to Nonlinear Analysis. MeGraw-Hill, New York.
3. Duren, P.L., 1970. Theory of Hp Spaces. Academic Press.
4. Fox, C., 1928. The asymptotic expansion of the generalized hypergeometric function. J. London Math. Soc., 2: 389-400.
5. Ruscheweyh, S., 1982. Convolutions in Geometric Function Theory. Sem. Math. Sup., University of Montreal Press.
6. Ruscheweyh, S., 1992. Geometric Properties of Cesáro Means. Results Math., 22: 739-748.
7. Ruscheweyh, S. and L. Salinas, 1993. Subordination by Cesáro Means. Complex Var. Theor. Appl., 21: 279-285.
8. Splina, L.T., 2008. On certain applications of the hadamard product. App. Math. Comp., 199: 653-662.
9. Slater, L.J., 1966. Generalized Hypergeometric Functions. Cambridge University Press, London.
10. Wright, E.M., 1935. The asymptotic expansion of the generalized hypergeometric function. J. London Math. Soc., 10: 286-293.
11. Wright, E.M., 1940. The asymptotic expansion of the generalized hypergeometric function. J. London Math. Soc., 46: 389-408.