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Graph Folding of Some Special Graphs

E. El-Kholy and A. El-Esawy
Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

Abstract: In this study we introduced the definition of graph folding then, we proved that we cannot fold any complete graph except the bipartite graph. By using incidence matrices we described the graph folding.

Key words: Tree, Graph Folding, Complete Graph, Bipartite Graph, Incidence Matrices

INTRODUCTION

Any graph G, which has exactly one component, is called connected and any connected graph which has no loops is called a tree [1].

Let G_1 and G_2 be graphs and $f: G_1 \to G_2$ be a continuous function. Then f is called a graph map, if

(i) for each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$,

(ii) for each edge $e \in E(G_1)$, $\dim(f(e)) \le \dim(e)$.

We call a graph map $f: G_1 \to G_2$ a graph folding, if and only if f maps vertices to vertices and edges to edges, i.e., for each $v \in V(G_1)$, $f(v) \in V(G_2)$ and for each $e \in E(G_1)$, $f(e) \in E(G_2)$.

Note that if the vertices of an edge $e = (u, v) \in E(G_1)$ are mapped to the same vertex, then the edge e will collapse to this vertex and hence we cannot get a graph folding.

We denote the set of graph foldings between graphs G_1 and G_2 by $\eta(G_1, G_2)$ and the set of graph foldings of G_1 into itself by $\eta(G_1)$. In the case of a graph folding f the set of singularities, $\sum f$, consists of vertices only.

The graph folding is non trivial iff $\sum f = \phi$. In this case the $\text{no.}V(f(G_1)) \le \text{no.}V(G_1)$, also $\text{no.}E(f(G_1)) \le \text{no.}E(G_1)$.

Proposition 1: Any tree T can be folded into itself by a sequence of graph foldings onto an edge.

Proof: Let T be a tree, then if $\text{no.}\,V(T) = n$, then $\text{no.}\,E(T) = n-1$. Now, we can define a sequence of graph foldings $f_i: T \to T_i$, where $f_i(T) = T_i$ is a subgraph of T_i, $i \in I$. Each of f_i fold one edge, or more, until we get only one edge which can not be folded any more.

Theorem 1: Let G be a complete graph, then there is no non-trivial graph foldings can be defined for G.

Proof: Since G is a complete graph, then every pair of vertices is adjacent. Let $f: G \to G$ be any graph map. This map will represent a non-trivial graph folding f, f maps vertices to vertices and edges to edges such that $\text{no.}V(f(G)) \le \text{no.}\,V(G)$. Thus f will maps at least two vertices to the same vertex and hence the edge joining these two vertices will collapse to a vertex and consequently it cannot represent a graph folding.

Example 1: Let G be a complete graph with vertex-set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G)=\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$, (Fig. 1).

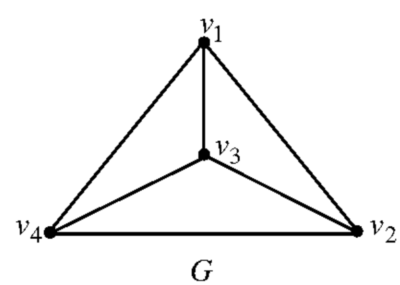

Fig. 1:G Complete Graph and f is not a Graph Folding

Let $f: G \to G$, be a graph map defined as follows:

$$f\{v_1, v_2, v_3, v_4\} = \{v_3, v_2, v_3, v_4\}$$

$$f\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$$
$$= \{(v_3, v_2), (v_3, v_3), (v_3, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$$

Then, this map is not a graph folding, because the image of $\{(v_1, v_3)\}$ is the vertex $\{v_3\}$.

From now when defining a graph map, a graph folding, any omitted vertex, edge, will be mapped onto itself. Thus, the map f in the above example can be redefined as follows:

$$f\{v_1\} = \{v_3\}, \quad f\{(v_1, v_3)\} = \{v_3\}$$

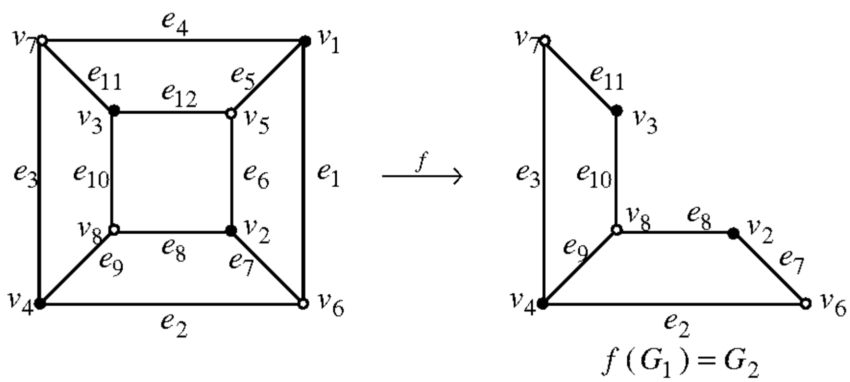

Fig. 2: G_1 Bipartite Graph and *f* is a Graph Folding

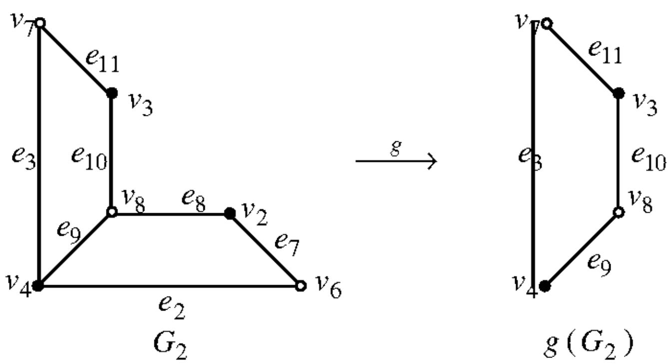

Fig. 3: G_2 Bipartite Graph and *g* is a Graph Folding

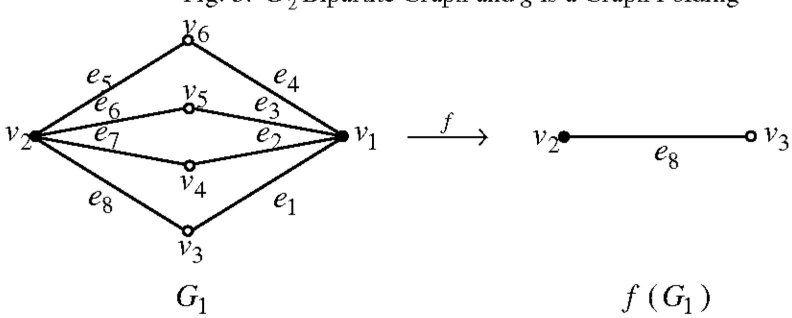

Fig. 4: G_1 Complete Bipartite Graph and *f* is a Graph Folding

Theorem 2: Any bipartite graph *G*, can be folded.

Proof: Let *G* be a bipartite graph. Then the vertex set $V(G)=\{v_1, v_2, \ldots, v_{r_1}, v_{r_1+1}, \ldots, v_r\}$ of *G* can be split into two sets $A=\{v_1, \ldots, v_{r_1}\}$ and $B=\{v_{r_1+1}, \ldots, v_r\}$, such that each edge of the graph joins a vertex in *A* to a vertex in *B*, [2]. Let $f: G \to G$ be a graph map, such that *f* maps vertices of *A* into another of *A* and vertices of *B* into another of *B*. Thus each edge $e=(v_i, v_j)$, $i=1, \ldots, r_1$, $j=r_1+1, \ldots r$, will be mapped to an edge $f(e)=(f(v_i), f(v_j))$, where, $f(v_i) \in V(A)$, $f(v_j) \in V(B)$ and hence *f* is a graph folding.

Example 2: Let G_1 be a bipartite graph, whose vertex sets $A=\{v_1, v_2, v_3, v_4\}$ and $B=\{v_5, v_6, v_7, v_8\}$, (Fig. 2). Let $f: G_1 \to G_1$, be graph folding defined as follows: $f\{v_1, v_5\}=\{v_4, v_8\}$ and $f\{e_1, e_4, e_5, e_6, e_{12}\}=\{e_2, e_3, e_9, e_8, e_{10}\}$.

Note that the graph $f(G_1)=G_2$ can be folded again as follows. Let $g: G_2 \to G_2$, be given by: $g\{v_2, v_6\}=\{v_3, v_7\}$, and $g\{e_2, e_7, e_8\}=\{e_3, e_{11}, e_{10}\}$, (Fig. 3).

Once, again g(G_2) can be folded until we get a graph consists of an edge

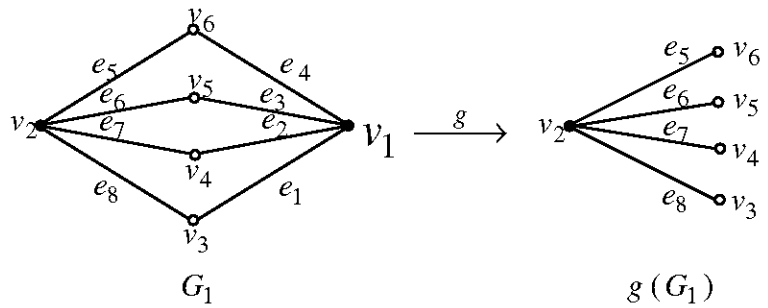

Fig. 5: G_1 Complete Bipartite Graph and g is a Graph Folding

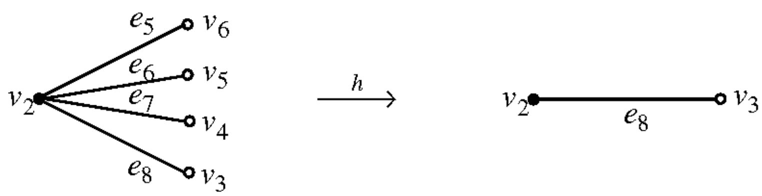

Fig. 6: G_2 Complete Bipartite and Graph h is a Graph Folding

Theorem 3: Any complete bipartite graph G, can be folded to an edge.

Proof: Let G be a complete bipartite graph with vertex set $V(G)=\{v_1, v_2, ..., v_{r_1}, v_{r_1+1}, ..., v_r\}$. This set again can be split into two sets $A=\{v_1, ..., v_{r_1}\}$ and $B=\{v_{r_1+1}, ..., v_r\}$ such that each vertex of A is joined to each vertex of B by exactly one edge, [2]. Thus,

$$E(G)=\{(v_1, v_{r_1+1}), (v_1, v_{r_1+2}), ..., (v_1, v_r), (v_2, v_{r_1+1}), ..., (v_2, v_r), ..., (v_{r_1}, v_{r_1+1}), (v_{r_1}, v_{r_1+2}), ..., (v_{r_1}, v_r)\}$$

Now, let $f: G \to G$, be a graph map defined by:

$$f(v_k)=\begin{cases} v_1, & \text{if} \quad k=1, ..., r_1 \\ v_{r_1+1}, & \text{if} \quad k=r_1+1, ..., r. \end{cases}$$

Thus the image of any edge of $E(G)$ will be the edge (v_1, v_{r_1+1}). Of course, this map is a graph folding.

Example 3: Let G_1 be a complete bipartite graph, whose vertex sets $A=\{v_1, v_2\}$ and $B=\{v_3, v_4, v_5, v_6\}$, (Fig. 4). Let $f: G_1 \to G_1$, be a graph map defined by:

$f\{v_1\}=\{v_2\}$, $f\{v_i\}=\{v_3\}$, $i=4, 5, 6, 7$, and $f\{e_i\}=\{e_8\}$, $i=1, ..., 8$.

It is clear that, this graph map is a graph folding with image the edge $e_8=(v_2, v_3)$.

Note that we can first fold G_1 by mapping $V(B)$ to itself and $V(A)$ to a vertex. Then, we can fold again by mapping $V(B)$ to a vertex. The composition will represent a graph folding to an edge, e.g., let $g: G_1 \to G_1$, be given by:

$g\{v_1\}=\{v_2\}$, and $g\{e_1, e_2, e_3, e_4\}=\{e_8, e_7, e_6, e_5\}$.

Thus, $g(G_1)=G_2$ is a complete bipartite graph such that $V(A')=\{v_2\}$ and $V(B')=\{v_3, v_4, v_5, v_6\}$, (Fig. 5). Let $h: G_2 \to G_2$, be a graph folding defined by:

$h\{v_4, v_5, v_6\}=\{v_3, v_3, v_3\}$ and $h\{e_j\}=e_8$, $j=5, 6, 7$.

It is clear that the composition, $h \circ g$, is a graph folding onto an edge.

Graph Folding and Incidence Matrices: Let G_1 be a finite graph, with the set of vertices $V(G_1)=\{v_1, ..., v_r\}$ and the set of edges $E(G_1)=\{e_1, ..., e_s\}$. Let $f \in \eta(G_1)$, such that $f(G_1) \neq G_1$. The incidence matrix denoted by $I=(\lambda_{kd})$ is defined by:

$\lambda_{kd} = 1$, if $v_k,\ k = 1, \ldots, r$ is a face of $e_d,\ d = 1, \ldots, s$ in G_1,

$\lambda_{kd} = 0$, if $v_k,\ k = 1, \ldots, r$ is not a face of $e_d,\ d = 1, \ldots, s$ in G_1.

The matrix I is of order $s \times r$, where s is the number of edges of G_1 and r is the number of vertices of G_1.

Let G_1 and G_2 be finite graphs and $f \in \eta(G_1, G_2)$. Then $f(G_1)$ is a subgraph of G_2. In particular; if $f \in \eta(G_1)$ with $f(G_1) = G_1' \neq G_1$. Then G_1' is a subgraph of G_1. This suggests that the incidence matrix I' of $f(G_1) = G_1'$ is a submatrix of the incidence matrix I of G_1 possibly after rearranging it's rows and columns.

We claim that the matrix I can be partition into four blocks, such that I' appears in the upper left corner block and a zero matrix O in the upper right one. The matrix R, the complement of I', will be a submatrix of I' possibly after deleting the rows and columns of I' which are not images of any of the edges $e_{s_1+1}, \ldots, e_s$ and the vertices $v_{r_1+1}, \ldots, v_r$, respectively.

The zero matrix O is due to the fact that non of the vertices $v_{r_1+1}, \ldots, v_r$ is incidence with any edge of the image.

$$I = \begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_{s_1} \\ e_{s_1+1} \\ e_{s_2+2} \\ \vdots \\ e_s \end{array} \begin{array}{c|c} v_1, v_2, \ldots v_{r_1} & v_{r_1+1}, \ldots, v_r \\ \hline I' & O \\ \hline Q & R \end{array}$$

Conversely, if the incidence matrix I of a graph G_1 can be partitioned into four blocks with a zero matrix at the right hand corner block. Then a graph folding may be defined, if there is any, as a map f of G_1 to an image $f(G_1)$ characterized by the incidence matrix I' which lie in the upper left corner of I. This map can be defined by mapping,

(i) the vertices $v_j,\ j = r_1 + 1, \ldots, r$ to the vertices $v_i,\ i = 1, \ldots, r_1$ if the j^{th} column in R is the same as the i^{th} column in I', after deleting the zeros from i^{th} column,

(ii) the edges $e_\nu,\ \nu = s_1 + 1, \ldots, s$ to the edges $e_\ell,\ \ell = 1, \ldots, s_1$ if e_ℓ and e_ν are incidence.

Examples 4: (a) Let G be a complete bipartite graph, whose vertex sets $A = \{v_1, v_3, v_5\}$ and $B = \{v_2, v_4, v_6\}$, respectively, (Fig. 7).

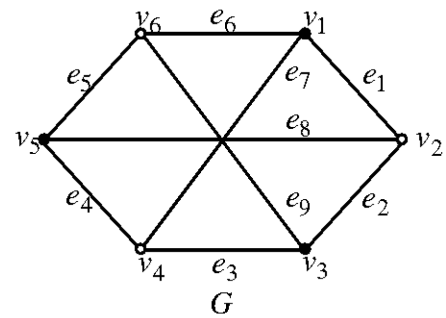

Fig. (7)

Fig. 7: G Complete Bipartite Graph

Now, the incidence matrix I of G has the following form:

$$I = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{array} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Now, we can partition I to take the following form

$$I = \begin{array}{c} \\ e_5 \\ e_6 \\ e_9 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_7 \\ e_8 \end{array} \begin{array}{cccc|cc} v_1 & v_3 & v_5 & v_6 & v_2 & v_4 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Thus, we can define a graph map $f : G \to G$ by:

$f\{v_2, v_4\} = \{v_6, v_6\}$and,

$f\{e_1, e_2, e_3, e_4, e_7, e_8\} = \{e_6, e_9, e_9, e_5, e_6, e_5\}$.

Thus, it is clear that this graph map is a graph folding, such that $f(G) = G'$ is a complete bipartite graph shown in Fig. 8 with incidence matrix I' appears in the upper left block of the above matrix, i.e.,

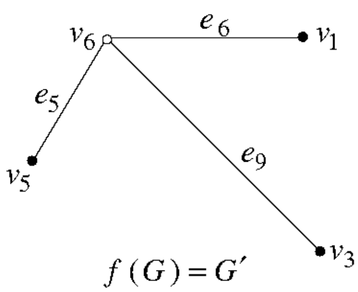

Fig. 8: G' Complete Bipartite Graph

$I' =$

	v_1	v_3	v_5	v_6
e_5	0	0	1	1
e_6	1	0	0	1
e_9	0	1	0	1

The matrix I' again can be partitioned as follows:

	v_3	v_6	v_1	v_5
e_9	1	1	0	0
e_5	0	1	0	1
e_6	0	1	1	0

Thus, we can define a graph map $g : G' \to G'$ by

$g\{v_1, v_5\} = \{v_3, v_3\}$ and $g\{e_i\} = e$, $i = 5, 6$.

Again the image, $g(G')$, is the graph shown in Fig. 9.

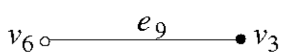

Fig. 9: Complete Bipartite Graph

Thus the complete bipartite graph G can be folded onto an edge.

(b) Let G be the complete graph given in Fig. 10.

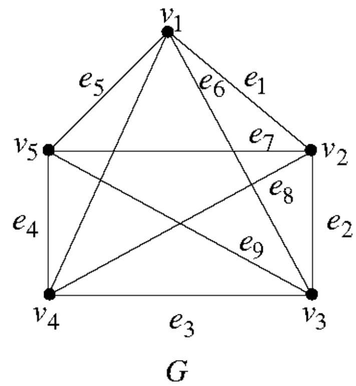

Fig. 10:G Complete Graph

The incidence matrix I *of* G is:

$I =$

	v_1	v_2	v_3	v_4	v_5
e_1	1	1	0	0	0
e_2	0	1	1	0	0
e_3	0	0	1	1	0
e_4	0	0	0	1	1
e_5	1	0	0	0	1
e_6	1	0	1	0	0
e_7	0	1	0	0	1
e_8	0	1	0	1	0
e_9	0	0	1	0	1
e_{10}	1	0	0	1	0

Now, if we partition I as follows:

	v_2	v_3	v_4	v_5	v_1
e_2	1	1	0	0	0
e_3	0	1	1	0	0
e_4	0	0	1	1	0
e_7	1	0	0	1	0
e_8	1	0	1	0	0
e_9	0	1	0	1	0
e_1	1	0	0	0	1
e_5	0	0	0	1	1
e_6	0	1	0	0	1
e_{10}	0	0	1	0	1

Then we cannot map v_1 to any of the vertices v_2, v_3, v_4 or v_5 and thus we cannot defined a graph folding. It is worth mentioning that no other partition of I will allow us to fold this graph, i.e., there is no graph folding of complete graphs.

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