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Introduction to the New Type of Algorithms for Accelerating Convergence of Sequence

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Abstract: A collection of new algorithms for accelerating the convergence of sequence of functions was described. The definitions and connections of these new algorithms with the improved functional epsilon algorithms are given. The effectiveness of these new algorithms was examined, namely the -algorithms, the -algorithms, the -algorithms, the -algorithms and the improved functional epsilon algorithms, for approximating solutions of a given power series. The estimates derived using the -algorithms are found to be substantially more accurate than other similar algorithms.

Key words: , , , -Algorithms, Improved Functional Epsilon Algorithm, Convergence Acceleration

INTRODUCTION

In this study, four new algorithms for accelerating the convergence of sequence of functions are introduced. We examine the effectiveness of these new algorithms by approximating the solutions of a given power series. These new algorithms are designed differently to the improved functional epsilon algorithm introduced in [1]. Basically, we express these new methods as a rational form. The elements of the numerator and the denominator are varied from algorithm to algorithm. Hence we shall see which algorithm produces a better approximation. The four algorithms described in this study are the most consistent algorithms found in our investigation. We have introduced the appropriate names for these new algorithms, namely the -algorithm, the -algorithm, the -algorithm and the -algorithm. The -algorithm was the first of the kind and this was further improved by the -algorithm and then -algorithm. Finally the -algorithm was developed and this is shown to be a good alternative to the improved functional epsilon algorithm. The prime motive for developing these new algorithms was to accelerate the convergence of sequence of functions and to find a better accelerator than the improved functional epsilon algorithm. In order to construct these new methods a similar procedure was used as to the improved functional epsilon algorithm, thus the following proposition is essential.

Proposition: The functional sequence used in each of the algorithms is based on the generating function $f(x, \lambda)$, which is a series of functions expressed as:

$$f(x,\lambda)=\sum_{i=0}^{\infty} C_i(x)\,\lambda^i, \tag{1}$$

in which $C_i(x) \in L_2[a, b]$ are given and $[a, b]$ is the domain of definition of $C_i(x)$ in some natural sense. We also suppose that $f(x, \lambda)$ is holomorphic as a function of λ at the origin $\lambda = 0$. Then (1) converges for values of $|\lambda|$ which are sufficiently small. In this study, we illustrate how these new algorithms can be used to accelerate the convergence of a series having the form (1) for $\lambda=1$.

The -algorithm: We define the α-algorithm of type (n, k) as:

$$\alpha(n,k;x)=\frac{\sum_{i=0}^{k}(-1)^i\left(\alpha(n+i+1,k-1;x)\right)\left(\int_a^b \Delta\alpha(n+i,k-1;x)\,dx\right)^{-1}}{\sum_{i=0}^{k}(-1)^i\left(\int_a^b \Delta\alpha(n+i,k-1;x)\,dx\right)^{-1}} \tag{2}$$

provided that the denominator of (2) is not equal to zero and the initial estimate given as:

$\alpha(n,0;x)=\sum_{i=0}^{n} C_i(x)$, for $n,k\in\aleph$. The operates, now and in sequel, on the variable n, for example:

$$\Delta\alpha(n-1,k)=\alpha(n,k)-\alpha(n-1,k)\,. \tag{3}$$

The -algorithm: The -algorithm is actually an improvement of the α-algorithm. Simply, we introduce the elements of $\binom{k}{i}$ and $(n+k+i)^{k-1}$ in the numerator and denominator of (2) and we find that the precision is increased substantially. We define the -algorithm of type (n, k) as:

$$\beta(n,k;x)=\frac{\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(n+k+i)^{k-1}(\beta(n+i+1,k-1;x))\left(\int_{a}^{b}\Delta\beta(n+i,k-1;x)\,dx\right)^{-1}}{\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(n+k+i)^{k-1}\left(\int_{a}^{b}\Delta\beta(n+i,k-1;x)\,dx\right)^{-1}} \qquad (4)$$

provided that the denominator of (4) is not equal to zero and the initial estimate given by $\beta(n,0;x)=\sum_{i=0}^{n}C_i(x)$, for $n,k\in\aleph$.

The -algorithm: We explore further to express an algorithm, which shall produce a better estimate than the improved functional epsilon algorithm. Therefore we define the γ-algorithm of type (n, k) as:

$$\gamma(n,k;x)=\frac{\sum_{i=0}^{k}(-1)^{i}(\gamma(n+i+1,k-1;x))\left(\int_{a}^{b}\Delta\gamma(n+k,k-1;x)\Delta\gamma(n+i,k-1;x)\,dx\right)^{-1}}{\sum_{i=0}^{k}(-1)^{i}\left(\int_{a}^{b}\Delta\gamma(n+k+i,k-1;x)\Delta\gamma(n+i,k-1;x)\,dx\right)^{-1}} \qquad (5)$$

provided that the denominator of (5) is not equal to zero and the initial estimate given by $\gamma(n,0;x)=\sum_{i=0}^{n}C_i(x)$, for $n,k\in\aleph$. In this particular algorithm two different types of difference component are used to improve the accuracy of the approximate solution.

The -algorithm: The -algorithm is actually an improvement of the -algorithm. As before, we multiply the integral component by $\binom{k}{i}$ and $(n+k+i)^{k-1}$ in the numerator and denominator of (5) and we find that the precision is better than previous three algorithms. We define the δ-algorithm of type (n, k) as:

$$\delta(n,k;x)=\frac{\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(n+k+i)^{k-1}(\delta(n+i+1,k-1;x))\left(\int_{a}^{b}\Delta\delta(n+k,k-1;x)\Delta\delta(n+i,k-1;x)\,dx\right)^{-1}}{\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(n+k+i)^{k-1}\left(\int_{a}^{b}\Delta\delta(n+k,k-1;x)\Delta\delta(n+i,k-1;x)\,dx\right)^{-1}} \qquad (6)$$

provided that the denominator of (6) is not equal to zero and the initial estimate given by $\delta(n,0;x)=\sum_{i=0}^{n}C_i(x)$, for $n,k\in\aleph$.

These new algorithms of type (n, k; x) can be laid out in a table:

$$\begin{matrix}(1,1;x) & (1,2;x) & (1,3;x) & \cdots\\ (2,1;x) & (2,2;x) & (2,3;x) & \cdots\\ (3,1;x) & (3,2;x) & (3,3;x) & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{matrix} \qquad (7)$$

where, n is the row sequence and k represents the column sequence.

The improved functional epsilon algorithm ( ): The improved functional epsilon algorithm is actually based on the integral Padé approximant [1-6]. The -algorithm is very efficient as an algorithm and is a successor of the classical functional epsilon algorithm [1, 5, 6].

We shall state the essential formula used in calculating the approximate solution

$$\varepsilon(n-1,k;x)=\varepsilon(n,k-1;x)-\left[\frac{\Delta\varepsilon(n,k-1;x)\int_{a}^{b}\Delta\varepsilon(n,k-1;x)\Delta\varepsilon(n-1,k-1;x)dx}{\int_{a}^{b}\Delta\varepsilon(n,k-1;x)\Delta^{2}\varepsilon(n-1,k-1;x)dx}\right] \qquad (8)$$

provided that the denominator of (8) is not equal to zero and the initial estimate given by $\varepsilon(n,0;x)=\sum_{i=0}^{n} C_i(x)$, for $n,k \in \aleph$.

Equivalence of the Estimates: Here we shall demonstrate the similarities of two groups. First of the similarity is between the three algorithms, namely, the -algorithm, the -algorithm and the -algorithm. We shall observe how these three algorithms produce similar expression of the approximating solution. The second group of similarity is between the -algorithm and the -algorithm. For this particular case we shall justify that the -algorithm reduces to the -algorithm, hence both algorithms are identical. In both of the groups we shall demonstrate the formula of each of the algorithms produce identical equations for a particular type of row sequence.
First we begin by expanding (4), the formula of the γ-algorithm of type (n, 1)

$$\gamma(n,1;x)=\frac{\dfrac{\gamma(n+1,0;x)}{\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}-\dfrac{\gamma(n+2,0;x)}{\int_a^b (\Delta\gamma(n+1,0;x))^2 dx}}{\dfrac{1}{\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}-\dfrac{1}{\int_a^b (\Delta\gamma(n+1,0;x))^2 dx}} \tag{9}$$

Multiplying the numerator and denominator of (9) by:

$$\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx \int_a^b (\Delta\gamma(n+1,0;x))^2 dx, \tag{10}$$

we obtain,

$$\gamma(n,1;x)=\frac{\gamma(n+1,0;x)\int_a^b (\Delta\gamma(n+1,0;x))^2 dx - \gamma(n+2,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}{\int_a^b (\Delta\gamma(n+1,0;x))^2 dx - \int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx} \tag{11}$$

To simplify the expression (11) further we must insert a component of $\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx$ in the numerator of (11), we get,

$$\begin{aligned}\gamma(n,1;x)=\Big[&\gamma(n+1,0;x)\int_a^b (\Delta\gamma(n+1,0;x))^2 dx\\ &-\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\\ &+\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\\ &-\gamma(n+2,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\Big]\div\\ &\Big[\int_a^b (\Delta\gamma(n+1,0;x))^2 dx-\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\Big]\end{aligned} \tag{12}$$

.

Collecting appropriate terms in the numerator of (12), we get,

$$\begin{aligned}\gamma(n,1;x)=\Big[&\gamma(n+1,0;x)\Big(\int_a^b (\Delta\gamma(n+1,0;x))^2 dx - \int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\Big)\\ &+\big(\gamma(n+2,0;x)-\gamma(n+1,0;x)\big)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\Big]\div\\ &\Big[\int_a^b (\Delta\gamma(n+1,0;x))^2 dx - \int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx\Big].\end{aligned} \tag{13}$$

Simplifying the expression (13) we obtain the similar formula

$$\gamma(n,1;x) = \gamma(n+1,0;x) - \frac{\Delta\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}{\int_a^b \left(\Delta\gamma(n+1,0;x)\right)^2 dx - \int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx} \tag{14}$$

To verify that the above equation of the -algorithm of type (n, 1) is identical to the -algorithm of type (n, 1) we must use the additive principle of integral in the denominator of (14), thus (14) becomes

$$\gamma(n,1;x) = \gamma(n+1,0;x) - \frac{\Delta\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}{\int_a^b \left[\left(\Delta\gamma(n+1,0;x)\right)^2 - \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)\right] dx} \tag{15}$$

Simplifying (15), we get,

$$\gamma(n,1;x) = \gamma(n+1,0;x) - \frac{\Delta\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}{\int_a^b \left[\Delta\gamma(n+1,0;x)\left\{\Delta\gamma(n+1,0;x) - \Delta\gamma(n,0;x)\right\}\right] dx} \tag{16}$$

Finally the -algorithm of type (n, 1) is expressed as:

$$\gamma(n,1;x) = \gamma(n+1,0;x) - \frac{\Delta\gamma(n+1,0;x)\int_a^b \Delta\gamma(n+1,0;x)\Delta\gamma(n,0;x)dx}{\int_a^b \left[\Delta\gamma(n+1,0;x)\Delta^2\gamma(n,0;x)\right] dx} \tag{17}$$

Similarly we obtain the expression of the ε-algorithm of type (n, 1) by simply inserting k=1 and n=n+1 in (8). We find that the -algorithm is identical to the ε-algorithm, given (17) and (8), respectively, for k=1.

Furthermore, it is relatively simple to show that the -algorithm of type (n, 1) is identical to the -algorithm of type (n, 1) and the -algorithm of type (n, 1). It can be seen that the formula of the -algorithm reduces to the formula of the algorithm when k=1. The reason for this is that the factors, $\binom{k}{i}$ and $(n+k+i)^{k-1}$, reduce to unity when k=1. Hence,

$$\binom{k}{i}(n+k+i)^{k-1} = 1 \tag{18}$$

for $n \in \aleph$ and $\iota \in [0, \kappa = 1]$.

Mathematically, we have demonstrated the fact that the three algorithms namely, the -algorithm of type (n, 1), the -algorithm of type (n, 1) and the -algorithm of type (n, 1) produce identical equations, that is

$$\gamma(n,1;x) \equiv \delta(n,1;x) \equiv \varepsilon(n,1;x) \tag{19}$$

for $n \in \aleph$ when k=1. Similarities between these methods is shown in Table 1 and 4.

The second group of similarity is between the -algorithm of type (n, 1) and the -algorithm of type (n, 1). It appears that the -algorithm of type (n, 1) reduces to the -algorithm of type (n, 1) for the same reason given by (16). Hence,

$$\alpha(n,1;\xi) \equiv \beta(n,1;\xi) \tag{20}$$

for $n \in \aleph$ when k=1. These similarities are demonstrated in the Table 1 and 4.

Applications of the New Algorithms: To demonstrate the performance of each of the new algorithms we take two familiar linear Fredholm integral equations of the second kind. We determine the consistency and stability of the results by examining the convergence of each of the algorithms for three particular types of row sequence. The findings are generalised by illustrating the effectiveness of these algorithms for approximating solution of a given power series solution. Consequently, we shall demonstrate the efficiency of these new algorithms and the -algorithm by showing the error obtained by each of the algorithms. We illustrate the convergence of the algorithms described by making three distinct comparisons of the estimates based on three particular types of row sequence. The first row sequence shows the similarities of the appropriate algorithms. Further two-row sequence illustrates the efficiency of each of the algorithms. In each case, the comparisons with other algorithms were made using a similar amount of data, that is using the same number of terms of the Neumann series. Also the errors listed in the tables are of absolute value.

Numerical Example 1: We investigate the convergence of functional sequences of the Neumann series solution of the linear integral equation. We shall consider the linear Fredholm integral equation of the form

$$f(x,\lambda) = g(x) + \lambda \int_0^1 k(x,y)\, f(y,\lambda)\, dy \tag{21}$$

where:

$$g(x) = x \quad \text{and} \quad k(x,y) = \begin{cases} (y-3)(1+x) & 0 \le y \le x \le 1, \\ (x-3)(1+y) & 0 \le x \le y \le 1. \end{cases}$$

This integral equation is a linear inhomogeneous Fredholm of the second kind with a non-degenerate kernel. The analytic solution of (21) is given by

$$f(x,\lambda) = \frac{3\left[\sinh(\omega x) + \omega\cosh(\omega x)\right] + \sinh(\omega x - \omega) - 2\omega\cosh(\omega x - \omega)}{(1+2\omega^2)\sinh(\omega) - 3\omega\cosh(\omega)} \tag{22}$$

where, $\omega = 2\sqrt{\lambda}$.

For a particular value =1 the analytic solution (22) in power series is,

$$f(x,1) = -0.229569 + 0.770431x - 0.459138x^2 + 0.513621x^3 - 0.153046x^4 \cdots \tag{23}$$

It is familiar that the Neumann series of (21) converges [7] and the first few terms of this series are,

$$f(x,\lambda) = \sum_{i=0}^{\infty} C_i(x)\,\lambda^i = x + (\tfrac{2}{3}x^3 - \tfrac{7}{6}x - \tfrac{7}{6})\lambda + (\tfrac{2}{15}x^5 - \tfrac{7}{9}x^3 - \tfrac{7}{3}x^2 + \tfrac{176}{45}x + \tfrac{176}{45})\lambda^2 + \cdots \tag{24}$$

In Table 1, we show the errors incurred by the α-algorithm of type (2, 1) with corresponding estimates derived from the -algorithm of type (2, 1), the -algorithm of type (2, 1), the -algorithm of type (2, 1) and the -algorithm of type (2, 1) for x=0(0.25)1. We observe the similarities of the two groups and find that there is no significant difference in the precision of the approximate solutions between the two groups.

Table 1: Errors Occurring in the Solution of (21) by the Five Algorithms Described

x	(2,1;x)= (2,1;x)	(2,1;x)= (2, 1;x)= (2,1;x)
0.0	0.001564	0.001732
0.25	0.001268	0.001471
0.50	0.000099	0.000122
0.75	0.001540	0.001318
1.0	0.001988	0.001784

For the five algorithms described we list the following approximate solutions used in calculating the errors displayed in Table 1:

$$\left.\begin{array}{l}\alpha(2,1;x)\\ \beta(2,1;x)\end{array}\right\} = \begin{array}{l}0.00017214x^9 + 0.0090835x^7 - 0.025304x^6 + 0.10502x^5 - 0.14156x^4 \\ +0.51001x^3 - 0.46996x^2 + 0.77199x - 0.22801\end{array} \tag{25}$$

$$\left.\begin{array}{l}\gamma(2,1;x)\\ \delta(2,1;x)\\ \varepsilon(2,1;x)\end{array}\right\} = \begin{array}{l}0.00017214x^9+0.0090835x^7-0.025305x^6+0.10502x^5-0.14155x^4\\ \quad +0.50998x^3-0.47007x^2+0.77216x-0.22784\end{array} \quad (26)$$

In Table 2, we show the errors incurred by the α-algorithm of type (1, 2) with corresponding estimates derived from the -algorithm of type (1, 2), the -algorithm of type (1, 2), the -algorithm of type (1, 2) and the -algorithm of type (2, 2) for x=0(0.25)1. We find that the precision of the -algorithm is similar to the -algorithm and is better than the other algorithms.

For the five algorithms described we list the following approximate solutions used in calculating the errors displayed in Table 2:

$$\begin{aligned}\alpha(1,2;x) = {} & 0.11323(-6)x^{13}+0.14345(-4)x^{11}-0.56672(-4)x^{10}+0.51703(-3)x^9\\ & -0.13550(-2)x^8+0.97335(-2)x^7-0.020754x^6+0.10289x^5\\ & -0.15223x^4+0.51336x^3-0.45991x^2+0.77054x-0.22946\end{aligned} \quad (27)$$

$$\begin{aligned}\beta(1,2;x) = {} & 0.10241(-6)x^{13}+0.13733(-4)x^{11}-0.51254(-4)x^{10}+0.52439(-3)x^9\\ & -0.14447(-2)x^8+0.97828(-2)x^7-0.020409x^6+0.10272x^5\\ & -0.15305x^4+0.51362x^3-0.45914x^2+0.77043x-0.22957\end{aligned} \quad (28)$$

$$\begin{aligned}\gamma(1,2;x) = {} & 0.11331(-6)x^{13}+0.14353(-4)x^{11}-0.56710(-4)x^{10}+0.51717(-3)x^9\\ & -0.13554(-2)x^8+0.97337(-2)x^7-0.020753x^6+0.10289x^5\\ & -0.15223x^4+0.51336x^3-0.45991x^2+0.77055x-0.22945\end{aligned} \quad (29)$$

$$\begin{aligned}\delta(1,2;x) = {} & 0.10247(-6)x^{13}+0.13739(-4)x^{11}-0.56710(-4)x^{10}+0.52447(-3)x^9\\ & -0.14449(-2)x^8+0.97829(-2)x^7-0.020409x^6+0.10272x^5\\ & -0.15305x^4+0.51362x^3-0.45914x^2+0.77043x-0.22957\end{aligned} \quad (30)$$

$$\begin{aligned}\varepsilon(2,2;x) = {} & 0.12192(-6)x^{13}+0.15444(-4)x^{11}-0.61023(-4)x^{10}+0.54341(-3)x^9\\ & -0.14585(-2)x^8+0.97832(-2)x^7-0.020407x^6+0.10272x^5\\ & -0.15305x^4+0.51362x^3-0.45914x^2+0.77043x-0.22957\end{aligned} \quad (31)$$

Table 2: Errors Occurring in the Solution of (21) by the Five Algorithms Described

x	(1,2;x)	(1,2;x)	(1,2;x)	(1,2;x)	(2,2;x)
0.0	0.111(-3)	0.109(-6)	0.122(-3)	0.759(-8)	0.168(-7)
0.25	0.899(-4)	0.963(-7)	0.104(-3)	0.258(-8)	0.111(-8)
0.50	0.701(-5)	0.142(-7)	0.863(-5)	0.234(-7)	0.211(-7)
0.75	0.109(-3)	0.109(-6)	0.931(-4)	0.918(-8)	0.222(-8)
1.0	0.141(-3)	0.165(-6)	0.126(-3)	0.137(-7)	0.191(-7)

It is tedious to list the approximate solutions of the five algorithms of type (1, 3). Consequently, in Table 3, we show the errors incurred by the -algorithm of type (1, 3) with corresponding estimates derived from the -algorithm of type (1, 3), the -algorithm of type (1, 3), the -algorithm of type (1, 3) and the -algorithm of type (4, 3) for x=0(0.25)1. We find that the precision of the δ algorithm is similar to the ε-algorithm and is better than the other algorithms.

Table 3: Errors Occurring in the Solution of (21) by the Five Algorithms Described

x	(1,3;x)	(1,3;x)	(1,3;x)	(1,3;x)	(4,3;x)
0.0	0.238(-9)	0.103(-12)	0.923(-10)	0.667(-14)	0.731(-14)
0.25	0.117(-9)	0.881(-13)	0.597(-11)	0.792(-14)	0.147(-15)
0.50	0.153(-9)	0.161(-13)	0.118(-9)	0.110(-13)	0.302(-14)
0.75	0.199(-9)	0.986(-13)	0.133(-10)	0.104(-13)	0.662(-14)
1.0	0.126(-9)	0.160(-12)	0.105(-9)	0.591(-14)	0.163(-14)

Numerical Example 2: We investigate the convergence of functional sequences of the Neumann series solution of the linear integral equation. We shall consider the linear Fredholm integral equation of the form

$$f(x,\lambda) = g(x) + \lambda \int_0^1 k(x,y)\, f(y,\lambda)\, dy \tag{32}$$

where:

$$g(x) = x^2 \quad \text{and} \quad k(x,y) = \begin{cases} y(2-x) & 0 \le y \le x \le 1, \\ x(2-y) & 0 \le x \le y \le 1. \end{cases}$$

This integral equation is also linear inhomogeneous Fredholm of the second kind with a non-degenerate kernel. The analytic solution of (32) is given by

$$f(x,\lambda) = \frac{1}{\lambda} - \frac{\cos(\mu x)}{\lambda} - \frac{[1-\cos(\mu)+\sin(\mu)\mu - 3\lambda]\sin(\mu x)}{\lambda \sin(\mu) + \lambda\mu\cos(\mu)} \tag{33}$$

where: $\mu = \sqrt{2\lambda}$.

For a particular value $\lambda = 1$ the analytic solution (33) in power series is,

$$f(x,1) = -0.229569 + 0.770431x - 0.459138x^2 + 0.513621x^3 - 0.153046x^4 \cdots \tag{34}$$

The first few terms of the Neumann series solution for (34), given by iteration of (32), are,

$$f(x,\lambda) = \sum_{i=0}^{\infty} C_i(x)\,\lambda^i = x^2 + \left[\tfrac{5}{12}x - \tfrac{1}{6}x^4\right]\lambda + \left[\tfrac{43}{180}x - \tfrac{5}{36}x^3 + \tfrac{1}{90}x^6\right]\lambda^2 + \cdots \tag{35}$$

In Table 4, we show the errors incurred by the α-algorithm of type (1, 1) with corresponding estimates derived from the -algorithm of type (1, 1), the -algorithm of type (1, 1), the -algorithm of type (1, 1) and the -algorithm of type (1, 1) for x=0(0.25)1. We observe the similarities of the two groups and find that there is no significant difference in the precision of the approximate solutions between the two groups.

For the five algorithms described we list the following approximate solutions used in calculating the errors displayed in Table 4:

$$\left.\begin{array}{l}\alpha(1,1;x)\\ \beta(1,1;x)\end{array}\right\} = \begin{array}{l}-0.77388(-3)x^8 + 0.011111x^6 + 0.027086x^5 - 0.16667x^4 \\ \quad -0.29418x^3 + x^2 + 0.88837x\end{array} \tag{36}$$

$$\left.\begin{array}{l}\gamma(1,1;x)\\ \delta(1,1;x)\\ \varepsilon(1,1;x)\end{array}\right\} = \begin{array}{l}-0.77190(-3)x^8 + 0.011111x^6 + 0.027017x^5 - 0.16667x^4 \\ \quad -0.29378x^3 + x^2 + 0.88777x\end{array} \tag{37}$$

Table 4: Errors Occurring in the Solution of (33) by the Five Algorithms Described

x	(1,2;x)= (1,2;x)	(1,2;x)= (1,2;x)= (1,2;x)
0.0	0	0
0.25	0.000024	0.000118
0.50	0.000166	0.000084
0.75	0.000357	0.000062
1.0	0.000390	0.000124

In Table 5, we show the errors incurred by the α-algorithm of type (1, 2) with corresponding estimates derived from the -algorithm of type (1, 2), the -algorithm of type (1, 2), the -algorithm of type (1, 2) and the -algorithm of type (2, 2) for x=0(0.25)1. We find that the precision of the algorithm is similar to the algorithm and is better than the other algorithms.

For the five algorithms described we list the following approximate solutions used in calculating the errors displayed in Table 5:

$$\alpha(1,2;x) = 0.30931(-8)x^{14} - 0.12113(-6)x^{12} - 0.70369(-6)x^{11} + 0.88825(-5)x^{10} + 0.38845(-4)x^{9} - 0.39683(-3)x^{8} - 0.14092(-2)x^{7} + 0.011111x^{6} + 0.029611x^{5} - 0.16667x^{4} - 0.29613x^{3} + x^{2} + 0.88838x \quad (38)$$

$$\beta(1,2;x) = 0.32818(-8)x^{14} - 0.11351(-6)x^{12} - 0.74660(-6)x^{11} - 0.88592(-5)x^{10} + 0.39151(-4)x^{9} - 0.39683(-3)x^{8} - 0.14101(-2)x^{7} + 0.011111x^{6} + 0.029613x^{5} - 0.16667x^{4} - 0.29613x^{3} + x^{2} + 0.88838x \quad (39)$$

$$\gamma(1,2;x) = 0.30910(-8)x^{14} - 0.12129(-6)x^{12} - 0.70320(-6)x^{11} + 0.88803(-5)x^{10} + 0.38852(-4)x^{9} - 0.39683(-3)x^{8} - 0.14092(-2)x^{7} + 0.011111x^{6} + 0.029611x^{5} - 0.16667x^{4} - 0.29613x^{3} + x^{2} + 0.88838x \quad (40)$$

$$\delta(1,2;x) = 0.32770(-8)x^{14} - 0.11377(-6)x^{12} - 0.74552(-6)x^{11} + 0.88577(-5)x^{10} + 0.39152(-4)x^{9} - 0.39683(-3)x^{8} - 0.14101(-2)x^{7} + 0.011111x^{6} + 0.029613x^{5} - 0.16667x^{4} - 0.29613x^{3} + x^{2} + 0.88838x \quad (41)$$

$$\varepsilon(2,2;x) = 0.31141(-8)x^{14} - 0.12220(-6)x^{12} - 0.70846(-6)x^{11} + 0.88183(-5)x^{10} + 0.39143(-4)x^{9} + 0.39683(-3)x^{8} - 0.14101(-2)x^{7} + 0.011111x^{6} + 0.029613x^{5} - 0.16667x^{4} - 0.29613x^{3} + x^{2} + 0.88838x \quad (42)$$

Table 5: Errors Occurring in the Solution of (33) by the Five Algorithms Described

x	(1,2;x)	(1,2;x)	(1,2;x)	(1,2;x)	(2,2;x)
0.0	0	0	0	0	0
0.25	0.170(-7)	0.644(-10)	0.731(-7)	0.128(-10)	0.977(-10)
0.50	0.110(-6)	0.672(-10)	0.498(-7)	0.152(-9)	0.131(-9)
0.75	0.230(-6)	0.582(-9)	0.399(-7)	0.145(-9)	0.858(-10)
1.0	0.248(-6)	0.843(-9)	0.765(-7)	0.120(-10)	0.941(-10)

As before, it is tedious to list the approximate solutions of the five algorithms of type (1, 3). Consequently, in Table 6 we show the errors incurred by the -algorithm of type (1, 3) with corresponding estimates derived from the -algorithm of type (1, 3), the -algorithm of type (1, 3), the -algorithm of type (1, 3) and the -algorithm of type (4, 3) for x=0(0.25)1. We find that the precision of the -algorithm is better than the other algorithms.

Table 6: Errors Occurring in the Solution of (33) by the Five Algorithms Described

x	(1,3;x)	(1,3;x)	(1,3;x)	(1,3;x)	(1,3;x)

0.0	0	0	0	0	0
0.25	0.111(-13)	0.194(-16)	0.433(-14)	0.243(-19)	0.128(-17)
0.50	0.352(-13)	0.504(-16)	0.581(-14)	0.645(-18)	0.150(-17)
0.75	0.412(-13)	0.265(-15)	0.383(-14)	0.494(-18)	0.236(-17)
1.0	0.306(-13)	0.361(-15)	0.414(-14)	0.945(-19)	0.188(-17)

CONCLUSION

In this study, we have shown four new algorithms, namely the -algorithm, the -algorithm, the -lgorithm and the -algorithm. These algorithms are essentially for accelerating the convergence of a sequence of functions. The prime motive of the development of these new algorithms was to accelerate convergence of sequence of functions. Moreover, the performance of these new algorithms has been demonstrated and compared with the -algorithm. Furthermore, we have demonstrated two groups of similarities. The first group of similarity is between the -algorithm, the -algorithm and the -algorithm. The second group of similarity is between the -algorithm and the -algorithm. The main purpose of demonstrating these new algorithms for three types of row sequence was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of each of the new algorithms. In all the numerical examples performed we have found that the -algorithm produces better estimates than the other similar algorithms and it may be considered to be a good alternative to the algorithm. Finally, an analytical investigation of these new algorithms is a subject of further research.

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