American Journal of Engineering and Applied Sciences

Original Research Paper

Analytical Solution Based on a New Series along with the Numeric Solution of the Non-Homogeneous Transient Heat Conduction Equation, in Hollow Cylinder under General Mixed Boundary Conditions

[1]Mehdi Zare and [2]F. Kardan

[1]Department of Mechanical Engineering, School of Engineering, Iran University of Science and Technology, Tehran, Iran
[2]Department of Mechanical Engineering, School of Engineering, Islamic Azad University, Maragheh Branch, Iran

Article history
Received: 30-11-2017
Revised: 11-04-2017
Accepted: 20-04-2018

Corresponding Author:
Mehdi Zare
Department of Mechanical Engineering, School of Engineering, Iran University of Science and Technology, Tehran, Iran
Email: Mehdi_zare@mecheng.iust.ac.ir

Abstract: In this study, the problem of transient conduction heat transfer for an infinite hollow cylinder under non-homogeneous mixed boundary conditions at the both surfaces simultaneously in radial direction with general heat source depend on time and radius, also general initial condition by the method of superposition and separation variables, is solved and temperature distribution is obtained analytically. A new series based on the Bessel functions is obtained for the problem of transient heat conduction without heat source by using separation of variables. Any function that has expanded conditions by the Fourier series can be expanded by this new series. Then, by expanding the heat source function according to this new series, the problem of transient heat transfer involving the thermal source has been solved and the radial temperature distribution is obtained. Due to the limited case studies, numerical solution of heat conduction equation with implicit finite difference method also is presented. Finally, a numerical example is given to compare between analytical and numerical solutions.

Keywords: Hollow Cylinder, Mixed Boundary Condition, Heat Conduction, Fourier Series, Finite Difference Method, Heat Source

Introduction

Heat transfer problems are prominent in engineering due to several applications in industry and environment performance of propulsion systems such as the design of conventional space and water heating systems, which in the cooling of electronic equipment, in the design of refrigeration and air-conditioning systems and in many manufacturing processes, as well plume and chemical nuclides dispersion, global warming, for example (Bejan and Kraus, 2003; Zare and Ganjalikhan nassab, 2014; Pletcher *et al*., 2012; Zalba *et al*., 2003). One of the important mechanisms of heat transfer is the conduction heat transfer. Heat conduction in cylindrical materials and tubes has been extensively studied due to various industrial applications such as thermodynamics, food processing, fuel cells, electrochemical reactors, air conditioning, high density microelectronics, composite materials, solidification processes, heat treatment of metals and many others (Dincer, 1995a; Dincer, 1994; Dincer, 1994; Carslaw and Jaeger, 1959). Chemical engineers encounter transient conduction in the cylindrical geometry when they analyze heat loss through pipe walls, heat transfer in double-pipe or shell-and-tube heat exchangers and other similar situations. The transient heat conduction problems include of heat source with mixed boundary conditions have some applications in engineering. In the nuclear reactors, cylindrical rods are heated internally by fission and are immersed in cooling fluid to produce energy using heat transfer at the surface. If the process by wetting or immersing in a fluid involves only part of the cylinder, a mixed boundary value problem is formed (Jackson, 2002). Analytical solutions of different linear heat conduction equations are meaningful in heat transfer theory. In addition, they are very useful to computational heat transfer to verify numerical analysis and to develop numerical schemes, grid generation methods. The common applied techniques for analytical methods to solve the heat conduction problem are Green functions, orthogonal expansions and the Laplace transformation (Carslaw and Jaeger, 1959). One of the effective methods for analytical solution of problem, including partial equations, is separation of variables. The partial differential equations are transferred into ordinary differential equations by separating the independent variables involved in the problem. A systematic procedure for determining the separation of variables for a given partial differential equation can be found in (Vedat, 1966; Adam *et al*., 2000). The superposition and the separation method are used in this study to get the analytical solutions of the temperature distribution. So far, many studies have been reported on conduction heat transfer in a hollow cylinder with different boundary conditions. (Holman, 2009) obtained the transient temperature field in a long solid cylinder, solid sphere and infinite flat plate, with a homogenous boundary condition. Exact analytical solution for Transient Heat Conduction in a Hollow Cylinder Using Duhamel Theorem was presented by Fazeli *et al*. (2013). Zhao *et al*. (2006) analyzed the temperature change when the thermal and thermo elastic properties are assumed to vary exponentially in the radial direction. Atefi *et al*. (2009) expressed an analytical solution of a two-dimensional temperature field in a hollow cylinder under a time periodic boundary condition using Fourier series. A number of analytical solutions for heat conduction can be obtained and found in the text book by Ozisik (1968). Recently, Wang and Liu (2013) have employed the method of separation of variables to develop the analytical solution of transient temperature fields for two dimensional transient heat conduction in a fiber-reinforced multilayer cylindrical composite. However, a cylindrical transient heat problem involved two non-homogeneous mixed boundary conditions simultaneously with general heat source and general initial condition not solved in the existing literature on this subject. In this study, the superposition and the separation method are used to get the analytical solutions of the temperature distribution. Initially, using these two methods, the heat transfer problem is solved, regardless of the heat source. Then, a new series will be obtained based on the Bessel functions. By expanding the heat source function based on this new series, an analytical solution to the main problem is found. Due to the lack of a similar analytical solution, for the purpose of comparison, the same problem is solved using the numerical method of implicit finite difference. Finally, a numeric example is solved with both methods and the results are compared. The results of these two comparisons show a very small difference between the results of analytic and numerical solutions.

Problem Formulation

The energy equation in cylindrical coordinates including radial conduction heat transfer and heat source in transient state with homogenous properties, is as follows:

$$T_{rr}(r,t)+\frac{1}{r}T_r(r,t)+\frac{q''(r,t)}{K}=\frac{1}{\alpha}T_t(r,t) \quad a\le r\le b \qquad t>0 \tag{1}$$

where, $q''(r,t)$ is the general heat source per unit volume inside the circular hollow cylinder, K is the thermal conductivity, α is the thermal diffusivity, r is the space variable, t is the time variable and a and b denote inner and outer radii, respectively. To simplify, $Q''(r,t)=\frac{q''(r,t)}{K}$ is considered.

Boundary conditions at the inner and outer surfaces are mixed and non-homogeneous. These conditions can be shown as follow:

$$\begin{aligned} T(a,t)+H_1T_r(a,t)&=G_1 \\ T(b,t)+H_2T_r(b,t)&=G_2 \end{aligned} \tag{2a}$$

$$H_i=\frac{K_i}{h_i} \quad i=1,2 \tag{2b}$$

where, K_1 and K_2 along with h_1 and h_2 are the thermal conductivities and the heat transfer coefficients at the inner and outer surfaces respectively. G_1 and G_2 is constant temperature of the surrounding medium at the inner and outer surfaces respectively. To better illustrate the physical concept, these two boundary conditions may be written as follows:

$$-K_1T_r(a,t)=h_1\left(T(a,t)-G_1\right) \tag{3a}$$

$$-K_2T_r(b,t)=h_2(T(b,t)-G_2) \tag{3b}$$

The concept of these two equations is that the heat flux on the internal and external surfaces are equal to the convection heat transfer between these surfaces and their surrounding medium.

The general initial condition is:

$$T(r,0)=I(r) \tag{4}$$

where, $I(r)$ is an initial temperature function.

Analytical Solution

By considering Equation (2) as a boundary condition, the problem cannot be solved directly. Therefore, the solution of the Equation (1) according to the superposition method can be considered as follows:

$$T(r,t)=T_0(r)+T_1(r,t) \tag{5}$$

where, $T_0(r)$ is the steady-state temperature and $T_1(r,t)$ is the transient-state temperature. Substituting (5) into

(1), (2) and (4), the differential heat conduction equation in steady-state is:

$$T_{0rr}(r)+\frac{1}{r}T_{0r}(r)=0 \tag{6}$$

And the boundary conditions of the steady-state are as follows:

$$T_0(a)+H_1T_{0r}(a)=G_1 \tag{7a}$$

$$T_0(b)+H_2T_{0r}(b)=G_2 \tag{7b}$$

The transient differential equation is:

$$T_{1rr}(r,t)+\frac{1}{r}T_{1r}(r,t)+Q''(r,t)=\frac{1}{\alpha}T_{1t}(r,t) \tag{8}$$

The conditions presented in Equations (9) and (10) must be satisfied:

$$T_1(a,t)+H_1T_{1r}(a,t)=0 \tag{9a}$$

$$T_1(b,t)+H_2T_{1r}(b,t)=0 \tag{9b}$$

$$T_1(r,0)=I(r)-T_0(r)=E(r) \tag{10}$$

Equation (6) is the Euler ordinary differential equation. With solving this equation in radial direction, the steady solution $T_0(r)$ is:

$$T_0(r)=A\,Ln(r)+B \tag{11}$$

where, A and B are two constants to be determined. From the boundary condition, Equation (7a) and (7b), Constants A and B get the following:

$$A=\left(\frac{G_2-G_1}{\Delta}\right) \tag{12a}$$

$$B=\frac{G_1\left(Ln(b)+\frac{H_2}{b}\right)-G_2\left(Ln(a)+\frac{H_1}{a}\right)}{\Delta} \tag{12b}$$

where, Δ is defined by:

$$\Delta=Ln\left(\frac{b}{a}\right)+\left(\frac{H_2}{b}-\frac{H_1}{a}\right)\neq 0 \tag{13}$$

Solution of Transient-State Problem

To solve the transient Equation (8), we first ignore the heat source term. Therefore:

$$T'_{1rr}(r,t)+\frac{1}{r}T'_{1r}(r,t)=\frac{1}{\alpha}T'_{1t}(r,t) \tag{14}$$

The boundary and initial condition are given by Equation (9) and (10).

Using the separation of variables, the solution of Equation (14) is considered as follows:

$$T'_1(r,t)=R(r).F(t) \tag{15}$$

Substituting (15) into (14), (9) and (10), the ordinary differential equation for variable of (r), is Bessel equation (Byerly, 2003) as follows:

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda^2r^2R=0 \tag{16}$$

$$R(a)+H_1\frac{dR}{dr}\Big|_{r=a}=0 \tag{17a}$$

$$R(b)+H_2\frac{dR}{dr}\Big|_{r=b}=0 \tag{17b}$$

where, λ^2 is separation constant. The general solution of Equation (15) is as follows (Boas, 2006):

$$R(r)=C_1J_0(\lambda r)+C_2Y_0(\lambda r) \tag{18}$$

where, $J_0(r)$ and $Y_0(r)$ and are Bessel functions of the first and second kind with order zero respectively (Boas, 2006). C_1 and C_2 are constants to be determined from the boundary conditions (17). Appling these boundary conditions:

$$\begin{aligned}&C_1\left[J_0(\lambda a)-H_1\lambda J_1(\lambda a)\right]\\&+C_2\left[Y_0(\lambda a)-H_1\lambda Y_1(\lambda a)\right]=0\end{aligned} \tag{19a}$$

$$\begin{aligned}&C_1\left[J_0(\lambda b)-H_2\lambda J_1(\lambda b)\right]\\&+C_2\left[Y_0(\lambda b)-H_2\lambda Y_1(\lambda b)\right]=0\end{aligned} \tag{19b}$$

For having nontrivial solution, the determinant of Equation (19) must be zero, thus the transcendental equation can be obtained as:

$$\begin{aligned}\omega(\lambda)=&\left[J_0(\lambda a)-H_1\lambda J_1(\lambda a)\right].\left[Y_0(\lambda b)-H_2\lambda Y_1(\lambda b)\right]\\&-\left[J_0(\lambda b)-H_2\lambda J_1(\lambda b)\right].\left[Y_0(\lambda a)-H_1\lambda Y_1(\lambda a)\right]=0\end{aligned} \tag{20}$$

Equation (20) in terms of λ, has infinite roots for the specified values of a, b, H_1 and H_2. Equation (20) can be rewritten as follows:

$$\omega(\lambda_j)=\left[J_0(\lambda_j a)-H_1\lambda_j J_1(\lambda_j a)\right].\left[Y_0(\lambda_j b)-H_2\lambda_j Y_1(\lambda_j b)\right]$$
$$-\left[J_0(\lambda_j b)-H_2\lambda_j J_1(\lambda_j b)\right].\left[Y_0(\lambda_j a)-H_1\lambda_j Y_1(\lambda_j a)\right] \quad (21)$$
$$=0\, j=1,2,3,\ldots$$

Here the eigenvalues λ_j (j = 1,2,3,…) are the j-th roots of the transcendental Equation (20).

The relationship between C_1, C_2 from Equation (19a) or (19b) is obtained:

$$C_2=-\frac{\left[J_0(\lambda_j b)-H_2\lambda_j J_1(\lambda_j b)\right]}{\left[Y_0(\lambda_j b)-H_2\lambda_j Y_1(\lambda_j b)\right]}C_1 \quad (22)$$

So, the solution in r direction can be written as follows:

$$R(r)=C_{1j}\varphi(\lambda_j r) \quad (23)$$

where, C_{1j} (j = 1, 2, 3,…) are constants and the Eigen functions $\varphi(\lambda_j r)$ is as follows:

$$\varphi(\lambda_j r)=\left[Y_0(\lambda_j b)-H_2\lambda_j Y_1(\lambda_j b)\right]J_0(\lambda_j r)$$
$$-\left[J_0(\lambda_j b)-H_2\lambda_j J_1(\lambda_j b)\right]Y_0(\lambda_j r) \quad (24)$$

Generally, homogeneous 2-point Boundary Value Problem (BVP) with homogeneous.

Boundary conditions have infinite number of solutions and the set of Eigen functions, form an orthogonal system with respect to the weight function r, over interval $a\le r\le b$ (Zill *et al*., 2011). Since Equation (16) and (17) are homogeneous 2-point boundary value problem with homogeneous boundary conditions, thus $\left\{\varphi(\lambda_j r)\right\}_{j=1}^{\infty}$ is the sequence of Eigen functions of a (BVP) on an interval $[a\ b]$ and can be written:

$$\int_a^b r.\varphi(\lambda_j r).\varphi(\lambda_k r)dr=0 \qquad j\ne k \quad (25)$$

Using this property, each piecewise smooth function like $f(r)$ in interval (a, b), can be expanded in terms of the Eigen functions $\varphi(\lambda_j r)$ as follows:

$$f(r)=\sum_{j=1}^{\infty}C_j.\varphi(\lambda_j r) \quad (26)$$

For obtaining C_j the both side of Equation (26) must be multiplied by $r.\varphi(\lambda_j r)$ and integrate from a to b:

$$C_j=\frac{\int_a^b r.f(r).\varphi(\lambda_j r).dr}{\int_a^b r.\varphi^2(\lambda_j r).dr} \quad (27)$$

Now, to find the solution $T_1(r,t)$ in Equation (8) we use the Eigen functions expansion method and assume the solution to be in the form:

$$T_1(r,t)=\sum_{j=1}^{\infty}T_{1j}(t).\varphi(\lambda_j r) \quad (28)$$

The heat source function is expanded in terms of Eigen functions as follows:

$$Q''(r,t)=\sum_{j=1}^{\infty}Q''_j\ (t).\varphi(\lambda_j r) \quad (29)$$

Substituting (28) and (29) into (8) yields the following equation:

$$\sum_{j=1}^{\infty}\left[\varphi_{rr}(\lambda_j r)+\frac{1}{r}\varphi_r(\lambda_j r)\right].T_{1j}(t)$$
$$+\left[Q''_j\ (t)-\frac{1}{\alpha}\dot{T}_{1j}(t)\right]\varphi(\lambda_j r)=0 \quad (30)$$

Using Equation (24), φ_{rr}, are calculated as follows:

$$\varphi_{rr}(\lambda_j r)=-\lambda_j^2.\varphi(\lambda_j r)-\frac{\varphi_r(\lambda_j r)}{r} \quad (31)$$

After substituting (31), into (30), we obtain the following characteristic equation:

$$\sum_{j=1}^{\infty}\left[-\frac{1}{\alpha}\dot{T}_{1j}(t)-\lambda_j^2.T_{1j}(t)+Q''_j\ (t)\right]\varphi(\lambda_j r)=0 \quad (32)$$

Thus:

$$-\frac{1}{\alpha}\dot{T}_{1j}(t)-\lambda_j^2.T_{1j}(t)+Q''_j\ (t)=0 \quad (33)$$

The initial condition for the nonhomogeneous ordinary differential Equation (33) can be calculated from Equation (10) and (28) as follows:

$$T_1(r,0)=E(r)=\sum_{j=1}^{\infty}T_{1j}(0).\varphi(\lambda_j r) \quad (34)$$

$T_{1j}(0)$ is given by the orthogonality condition:

$$T_{1j}(0)=\frac{\int_a^b r.E(r).\varphi(\lambda_j r).dr}{\int_a^b r.\varphi^2(\lambda_j r).dr} \quad (35)$$

The solution of Equation (33) with initial condition (35) is:

$$T_{1j}(t)=\int_0^t \alpha Q''_j\ (\bar{t}).e^{-\alpha\lambda_j^2(t-\bar{t})}d\bar{t}+T_{1j}(0).e^{-\alpha\lambda_j^2 t} \quad (36)$$

In Equation (36), $Q''_j\ (\tau)$ is unknown. Again, by the orthogonality condition and using Equation (29), we can write:

$$Q''_j\ (\tau)=\frac{\int_a^b \xi.Q''(\xi,\bar{t}).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi} \quad (37)$$

Plugging (37) into (36) the result is as follows:

$$T_{1j}(t)=\int_0^t \alpha\left[\frac{\int_a^b \xi.Q''(\xi,\bar{t}).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\alpha\lambda_j^2(t-\bar{t})}d\bar{t}+T_{1j}(0).e^{-\alpha\lambda_j^2 t} \tag{38}$$

Substituting (38) into (28), the transient-state solution is as follows:

$$T_1(r,t)=\sum_{j=1}^{\infty}\left\{\left(\int_0^t \alpha\left[\frac{\int_a^b \xi.Q''(\xi,\bar{t}).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\alpha\lambda_j^2(t-\bar{t})}d\bar{t}\right)+\left[\frac{\int_a^b \xi.E(\xi).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\alpha\lambda_j^2 t}\right\}.\varphi(\lambda_j r) \tag{39}$$

Finally, using the superposition principle in accordance with Equation (5) with the sum of the transient and steady-state solutions, the solution of the conduction heat transfer Equation (1) is as follows:

$$\begin{aligned}T(r,t)=&\sum_{j=1}^{\infty}\left\{\left(\int_0^t \alpha\left[\frac{\int_a^b \xi.Q''(\xi,\bar{t}).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\alpha\lambda_j^2(t-\bar{t})}d\bar{t}\right)+\left[\frac{\int_a^b \xi.E(\xi).\varphi(\lambda_j\xi).d\xi}{\int_a^b \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\alpha\lambda_j^2 t}\right\}.\varphi(\lambda_j r)\\&+\left(\frac{G_2-G_1}{\Delta}\right)Ln(r)+\frac{G_1\left(Ln(b)+\frac{H_2}{b}\right)-G_2\left(Ln(a)+\frac{H_1}{a}\right)}{\Delta}\quad \Delta\neq 0\end{aligned} \tag{40}$$

where, Δ is calculated from Equation (13).

To simplify the answer and to plot the result, Equation (40) is converted into dimensionless form, by assuming the dimensionless numbers given below:

$$\begin{aligned}&r^*=\frac{r}{b},\eta=\frac{a}{b},\theta=\frac{T}{T_r},\theta^*(r^*)=\frac{I(r)}{T_r},\tau=\frac{\alpha t}{b^2}\\&G_i^*=\frac{G_ih_ib}{K_iT_r}=\frac{G_iBi_i}{T_r},(i=1,2),\\&Bi_i=\frac{b}{H_i},G(r^*,\bar{t})=\frac{Q''(r,t).b^2}{T_r}\end{aligned} \tag{41}$$

where, r^*, η, θ,θ^*, T_r, τ, Bi, $G_i^*,\mathcal{G}$ are dimensionless radius, dimensionless thickness, dimensionless temperature, dimensionless initial temperature function, reference temperature Fourier number, Biot number, Dimensionless parameter in relation to G_i and dimensionless number of the heat source, respectively.

The boundary value problem of the heat conduction in dimensionless form is:

$$\theta_{r^*r^*}+\frac{1}{r^*}\theta_{r^*}+G=\theta_\tau \tag{42}$$

$$\theta(\eta,\tau)+\frac{1}{Bi_1}\frac{\partial\theta(\eta,\tau)}{\partial r^*}=\frac{G_1^*}{Bi_1} \tag{43a}$$

$$\theta(1,\tau)+\frac{1}{Bi_2}\frac{\partial\theta(1,\tau)}{\partial r^*}=\frac{G_2^*}{Bi_2} \tag{43b}$$

$$\theta(r^*,0)=\theta^*(r^*) \tag{43c}$$

Thus, the dimensionless Equation (40) is as follows:

$$\begin{aligned}\theta(r^*,\tau)=&\theta_0(r^*)\\&+\sum_{j=1}^{\infty}\left\{\begin{aligned}&\left(\int_0^\tau\left[\frac{\int_\eta^1 \xi.\mathcal{G}(\xi,\bar{t}).\varphi(\lambda_j\xi).d\xi}{\int_\eta^1 \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\lambda_j^2(\tau-\bar{t})}d\bar{t}\right)\\&+\left[\frac{\int_\eta^1 \xi.(\theta^*(\xi)-\theta_0(\xi)).\varphi(\lambda_j\xi).d\xi}{\int_\eta^1 \xi.\varphi^2(\lambda_j\xi).d\xi}\right].e^{-\lambda_j^2\tau}\end{aligned}\right\}.\varphi(\lambda_j r^*)\end{aligned} \tag{44}$$

where:

$$\begin{aligned}&\theta_0(r^*)\\&=\left(\frac{Bi_1.G_2^*-Bi_2.G_1^*}{Bi_1.Bi_2.\Delta^*}\right)Ln(r^*)+\left(\frac{1}{\Delta^*}\right)\\&\left[\left(\frac{G_2^*}{Bi_2^2}\right)-\left(\frac{G_2^*}{Bi_1}\right)\left(Ln(\eta)+\frac{1}{Bi_1.\eta}\right)\right],\Delta^*\neq 0\end{aligned} \tag{45}$$

$$\Delta^*=-Ln(\eta)+\left(\frac{1}{Bi_2}-\frac{1}{Bi_1.\eta}\right) \tag{46}$$

Numerical Solution

In the study of transient heat conduction in hollow cylinders under two non-homogeneous mixed boundary conditions simultaneously, with general internal heat source and initial condition, only limited studies can be found. To check the quality and accuracy of the analytical solution, in this section using the Finite Difference Method (FDM), numerical solution of the problem is also formulated. Then, a numerical example is given to compare analytical solution and numerical solution.

Using the implicit finite difference method, the dimensionless partial differential Equation (42) and conditions (43) are discrete to the algebraic equations. Introduce a mesh or nodes r_i^*, (i = 1,2, …, N) with uniform spacing Δr^*. Consider also a mesh in time formed by instants of time separated by a constant amount of time $\Delta\tau$. The mesh points in time are τ_1, τ_2, …τ_1, τ_{j+1},… τ_m. Note that time level t_1 represents the initial condition of the system. An implicit scheme is readily obtained by using a central differences formula for the space derivative and an Euler forward difference for the time derivative. Hence, writing $\theta\left(r_i^*,\tau_j\right)\approx\theta_{i,j}$, the FDM analog is:

$$\theta_{i,j}=\left[\left(r_i^*-\Delta r^*\right)\theta_{i-1,j+1}+\left(r_i^*+\Delta r^*\right)\theta_{i+1,j+1}-2\theta_{i,j+1}\right]\times\left(\frac{-\Delta\tau}{r_i^*\Delta r^{*2}}\right)+\theta_{i,j+1}-\Delta\tau.\mathcal{G}_{i,j+1}\tag{47}$$

where, $\mathcal{G}_{i,j+1}=\mathcal{G}\left(r_i^*,\tau_{j+1}\right)$. Equation (47) for the first node $\left(r_1^*=\eta\right)$ and the last node $\left(r_N^*=1\right)$ is written as follows:

$$\theta_{1,j}=\left[\left(\eta-\Delta r^*\right)\theta_{-2,j+1}+\left(\eta+\Delta r^*\right)\theta_{2,j+1}-2\theta_{1,j+1}\right]\times\left(\frac{-\Delta\tau}{\eta\Delta r^{*2}}\right)+\theta_{i,j+1}-\Delta r\mathcal{G}_{1,j+1}\tag{48}$$

$$\theta_{N,j}=\left[\left(1-\Delta r^*\right)\theta_{N-1,j+1}+\left(1+\Delta r^*\right)\theta_{N+1,j+1}-2\theta_{N,j+1}\right]\times\left(\frac{-\Delta\tau}{\Delta r^{*2}}\right)+\theta_{N,j+1}-\Delta\tau.\mathcal{G}_{N,j+1}\tag{49}$$

In Equation (48) and (49), the auxiliary node to the left of node 1 will be labeled node -2 and the one to the right of node N will be node N + 1. These nodes are fictitious (ghost node) they are only a device to obtain higher accuracy and do not appear in the final formulae. Second order accurate central difference formulae are derived by performing differential energy balances to approximate the corresponding boundary condition and the temperature of the ghost nodes is eliminated by combining the result with the finite difference formula corresponding to the boundary node. According to the boundary conditions in Equation (43a), a second order accurate finite difference approximation at (r^* = η) is:

$$\theta_{1,j+1}+\frac{1}{Bi_1}\left(\frac{\theta_{2,j+1}-\theta_{-2,j+1}}{2\Delta r^*}\right)=\frac{G_1^*}{Bi_1}\tag{50}$$

A similar approximation at at (r^* = 1) yields:

$$\theta_{N,j+1}+\frac{1}{Bi_2}\left(\frac{\theta_{N+1,j+1}-\theta_{N-1,j+1}}{2\Delta r^*}\right)=\frac{G_2^*}{Bi_2}\tag{51}$$

Combining with the finite difference approximation according to the Equation (48) and (49) and rearranging yields:

$$\theta_{1,j}=\left(\frac{-\Delta\tau}{\eta\Delta r^{*2}}\right)\left[2\eta\theta_{2,j+1}+\left(-2\eta\Delta r^*+\Delta r^{*2}\right)G_1^*\left(\tau_{j+1}\right)\right]+\left(\frac{2\Delta\tau}{\Delta r^{*2}}-\frac{2Bi_1\Delta\tau}{\Delta r^*}+\frac{Bi_1\Delta\tau}{\eta}+1\right)\theta_{1,j+1}-\Delta\tau.G_{1,j+1}\tag{52}$$

and:

$$\theta_{N,j}=\left(\frac{-\Delta\tau}{\Delta r^{*2}}\right)\left[2\theta_{N-1,j+1}+\left(2\Delta r^*+\Delta r^{*2}\right)G_2^*\left(\tau_{j+1}\right)\right]+\left(\frac{2\Delta\tau}{\Delta r^{*2}}+\frac{2\mathrm{Bi}_2\Delta\tau}{\Delta r^*}+\mathrm{Bi}_2\Delta\tau+1\right)\theta_{N,j+1}-\Delta\tau.G_{N,j+1}\tag{53}$$

Equation (47), (52) and (53) are a nice tri-diagonal system of algebraic equations. This set of algebraic equations can be solved by Thomas algorithm (El-Mikkawy and Moawwad, 2004).

Verification

Verification of the New Series

Because of the analytical transient- state solution of the transient heat transfer in hollow cylinder with mixed boundary conditions is dependent on the series given in Equation (26), here, with an example, the precision of this series is investigated. Firstly, to expansion a function by this series, the roots of Equation (21) must be obtained. A graph of Equation (21), is shown in Fig. 1.

For the solution region, [1, 10] and amounts of, H_1 = H_2 = 10. The first 21 roots of this equation are given in Table 1.

Let's assume that the function $f(r)$ to be expanded as in Equation (26) is the square function expressed as:

$$f\left(r\right)=\begin{Bmatrix}1 & for & 1\le r\le 5\\ -1 & for & 5<r\le 10\end{Bmatrix}\tag{54}$$

This function is piecewise smooth in solution region [1, 10]. Furthermore Both H_1 and H_2 are equal to 10 (m). The function expressed by Equation (54) and three approximate expansions are plotted in Fig. 2.

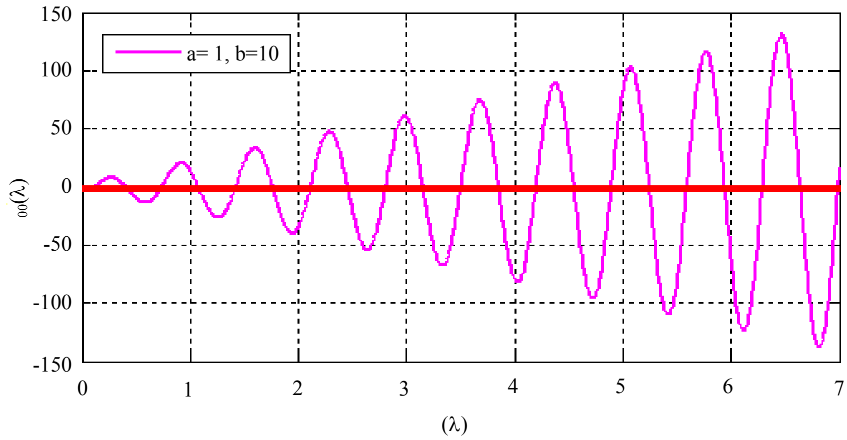

Fig. 1: Equation (20), as λ for $H_1 = H_2 = 10$

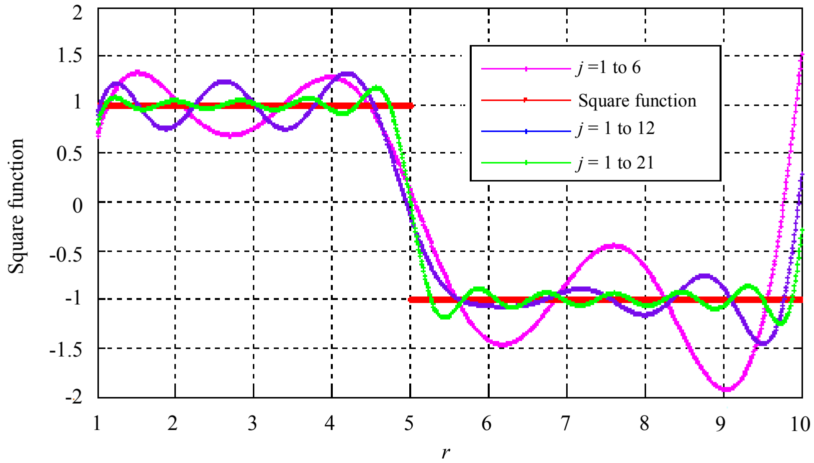

Fig. 2: Square function with Equation (46) and three approximated expansion With the number of different sentences, in [1, 10] and with $H_1 = H_2 = 10$

Table 1: First twenty-one roots of Equation 20, in solution region [1, 10] for $H_1 = H_2 = 10$

λ_j for $H_1 = 10$	$H_2 = 10$	$a = 1$ and $b = 10$
0.1121	2.4580	4.8945
0.4057	2.8054	5.2430
0.7369	3.1531	5.5917
1.0765	3.5011	5.9404
1.4197	3.8492	6.2891
1.7648	4.1975	6.6379
2.111	4.5460	6.9866

Table 2: First twenty-one C_j Square function with Equation (46) in [1,10] and with $H_1 = H_2 = 10$

j	C_j	j	C_j	j	C_j
1	-107.7978	8	-0.4516	15	-0.0406
2	-18.0603	9	-0.4908	16	-0.1962
3	-4.2623	10	-0.3128	17	-0.1096
4	-2.2454	11	-0.0660	18	-0.0072
5	-1.4590	12	-0.2569	19	-0.0640
6	-1.1704	13	-0.2431	20	-0.1444
7	-0.3566	14	-0.0814	21	-0.0267

It is obvious that, by increasing the number of series sentences fluctuation range around the main function is lower from the same radial domain. Summation over the first 21 terms produced an acceptable estimation in the interval [1,10] with some apparent oscillations around the exact function. The calculated C_j coefficients for this function are shown in Table 2.

The C_j coefficients have a general absolute level of <1 except C_1 to C_6. The coefficients are less than one, indicating that their associated terms in the series are very small.

Comparison between Analytical Solution and Numerical Solution

As seen, the analytical solution to the problem of the transient heat conduction in a hollow cylinder, including general heat source with non-homogeneous mixed boundary conditions at the both boundaries, is somewhat complicated. Thus, in this section, to better understanding of the analytical solution, a numerical example is considered. Then, analytical solution is compared with numerical solution.

Consider the transient heat conduction in a hollow cylinder with general heat source as follows:

$$Q''(r,t) = r.e^{-t} \tag{55}$$

According to the conditions of the problem, the dimensionless number of the heat source is as follows:

$$G(r^*,\tau) = 4.32\ r^*.e^{-36\tau} \tag{56}$$

The boundary conditions in dimensionless form are as follows:

$$\theta(\eta,\tau) + \frac{5}{3}\frac{\partial\theta(\eta,\tau)}{\partial r^*} = \frac{4}{5} \tag{57a}$$

$$\theta(1,\tau) + \frac{5}{6}\frac{\partial\theta(1,\tau)}{\partial r^*} = \frac{6}{5} \tag{57b}$$

According to the boundary conditions, can be writing:

$$Bi_1 = \frac{3}{5} \tag{58a}$$

$$Bi_2 = \frac{6}{5} \tag{58b}$$

Dimensionless initial condition is:

$$\theta(r^*,0) = 1 \tag{59}$$

Figure 3 depicts analytical and numerical dimensionless temperature profiles along the radial of the hollow cylinder at different times. To calculate the analytical radial dimensionless temperature distribution, twenty-one sentences from series are used in Equation (44) and in the numerical method, number of the mesh points in space are twenty-one.

In addition, the temperature distribution in steady-state is also shown in this graph. As can be seen, the distribution of numerical and analytical temperatures has little difference. It is clear that as the time pass, the temperature distribution in the cylinder increases and approaches to steady-state temperature. In addition, the rate of increase in temperature at the inner surface is higher than the outer surface, thus direction of heat transfer is, from the inner surface to the outer surface. In order to analyze the temperature distribution along the radial of the hollow cylinder, all three factors must be considered: initial condition, boundary conditions and heat source function. Despite the increase in radius, the power of the heat source increases, but the outer surface temperature is lower than the inner surface temperature. The reason for this behavior is the boundary conditions of the problem, because of $Bi_2 > Bi_1$, so the convection heat transfer from the outer surface is greater and the temperature drop in this area is higher. To show the effect of time on the temperature of the internal, middle and external surfaces, in Fig. 4, the analytical and numerical temperature variations of these surfaces are shown with time.

Furthermore, In Table 3, the numerical value of analytical temperature variations of these surfaces from the initial time to the steady state are given for several τ values. Obviously, the temperatures of these surfaces will increase over time.

In Fig. 5, a schematic of the geometry of the problem and the direction of heat transfer are shown.

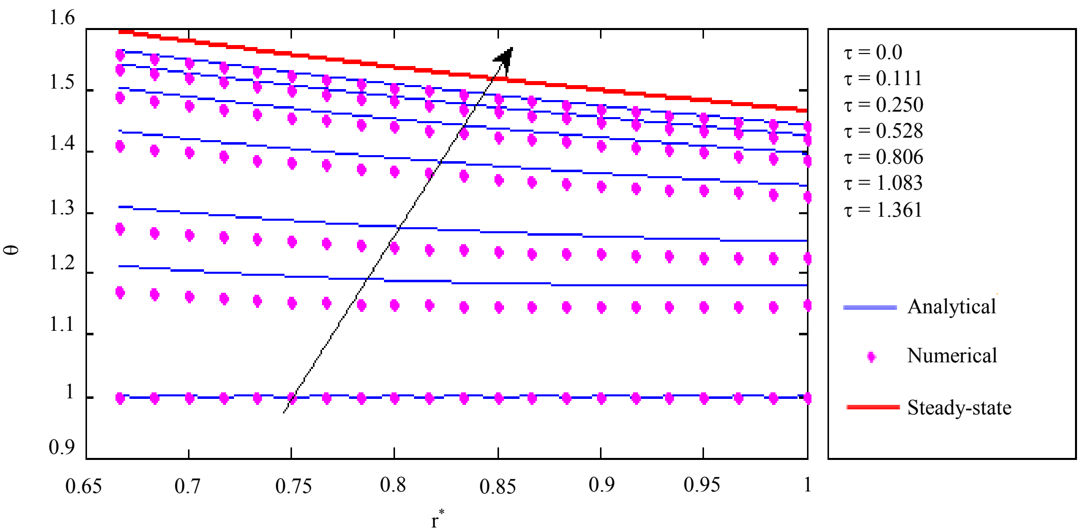

Fig. 3: Analytical and numerical dimensionless radial temperature variations in hollow cylinder from initial time to the steady state

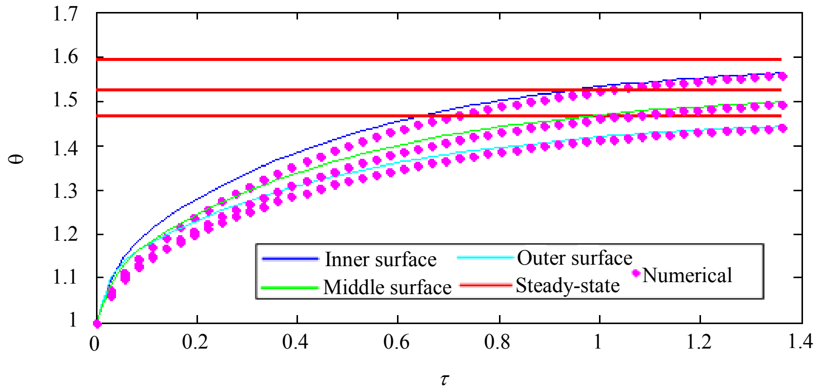

Fig. 4: Analytical and numerical dimensionless temperature variations at the inner, middle and outer surfaces of the hollow cylinder with Fourier number

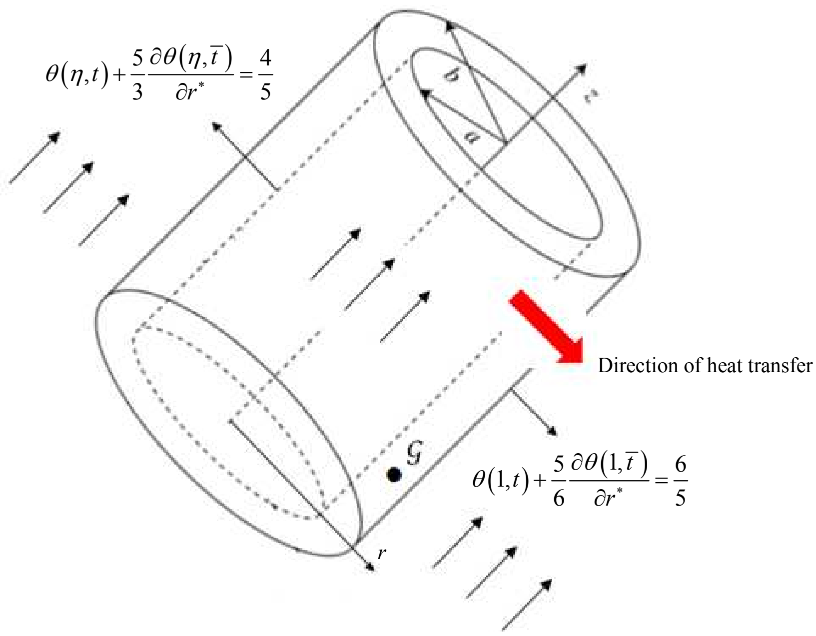

Fig. 5: The schematic of a hollow cylinder contains heat source, under non-homogeneous mixed boundary conditions in the both surfaces along with the direction of heat transfer

Table 3: Analytical dimensionless temperatures at the inner, middle and outer surfaces of the hollow cylinder at various dimensionless time

	r^*		
τ	0.67	0.83	1.0
0	1.000	1.000	1.000
0.111	1.212	1.186	1.181
0.25	1.309	1.27	1.252
0.389	1.379	1.333	1.304
0.528	1.432	1.38	1.344
0.667	1.471	1.415	1.374
0.805	1.501	1.442	1.396
0.944	1.524	1.524	1.413
1.083	1.541	1.477	1.426
1.222	1.554	1.488	1.435
1.361	1.563	1.497	1.442
$\tau\to\infty$	1.593	1.522	1.464

Conclusion

In this article, two analytical and numerical solutions were presented for the problem of transient heat conduction in hollow cylinder with general source term, under mixed non-homogeneous boundary condition at the inner and outer surfaces simultaneously. By solving the governing equation for conduction heat transfer, temperature distribution was obtained. An Analytical solution was expressed based on a new series. Using this series, each piecewise smooth function can be expanded. The quality of the series was examined with an example. Numerical solution was performed using implicit finite difference method. The solutions obtained from the analytic and numerical methods were compared with a numerical example. Due to the difference in the nature of these two methods, the resulting solutions differed slightly.

Acknowledgment

The authors wish to thank Dr. N. Nuri Professor mechanical Engineering at Iran University of Science and Technology.

Author's Contributions

All authors equally contributed in this work.

Ethics

This article is original. Author declares that are not ethical issues that may arise after the publication of this manuscript.

References

Adam, P., R.N. Nicoud, M. Bailly and O. Ludemann-Hombourger, 2000. Process and device for separation with variable-length. U.S. Patent No. 6,136,198.

Atefi, G., M.A. Abdous, A. Ganjehkaviri and N. Moalemi, 2009. An analytical solution of a two-dimensional temperature field in a hollow cylinder under a time periodic boundary condition using Fourier series. J. Mech. Eng. Sci., 223: 1889-1901. DOI: 10.1243/09544062JMES1382

Bejan, A. and A.D. Kraus, 2003. Heat Transfer Handbook. 1st Edn., John Wiley and Sons, New York, ISBN-10: 0471390151, pp: 1479.

Boas, M.L., 2006. Mathematical Methods in the Physical Sciences. 1st Edn., Wiley, New Delhi, ISBN-10: 8126508108, pp: 864

Byerly, W.E., 2003. An elementary treatise on Fourier's series and spherical, cylindrical and ellipsoidal harmonics, with applications to problems in mathematical.

Carslaw, H.S. and J.C. Jaeger, 1959. Heat in Solids. 1st Edn., Clarendon Press, Oxford.

Dincer, I., 1994. Transient temperature distributions in spherical and cylindrical food products subjected to hydro cooling. Int. J. Energy Res., 18: 741-749. DOI: 10.1002/er.4440180805

Dincer, I., 1995a. Transient heat transfer analysis in air cooling of individual spherical products. J. Food Eng., 26: 453-467. DOI: 10.1016/0260-8774(94)00067-J

Dincer, I., 1995b. Estimation of dimensionless temperature distributions in spherical products during hydro cooling. Int. Commun. Heat Mass Transfer, 22: 123-131. DOI: 10.1016/0735-1933(94)00058-S

El-Mikkawy, M.E.A. and Moawwad, 2004. A fast algorithm for evaluating nth order tri-diagonal determinants. J. Computat. Applied Math., 166: 581-584. DOI: 10.1016/j.cam.2003.08.044

Fazeli, H., M.A. Abdous, H. Karabi N. Moallemi and M. Esmaeili, 2013. Analysis of transient heat conduction in a hollow cylinder using Duhamel theorem. Int. J. Thermophys., 34: 350-365. DOI: 10.1007/s10765-013-1418-y

Holman, J.P., 2009. Heat Transfer. 10th Edn., McGraw-Hill Education, Boston, ISBN-10: 0073529362, pp: 752.

Jackson, J.D., 2002. Consideration of the heat transfer properties of supercritical pressure water in connection with the cooling of advanced nuclear reactors. Proceedings of the 13th Pacific Basin Nuclear Conference, Oct. 21-25, Beijing, China.

Ozisik, M.N., 1968. Heat conduction in the cylindrical coordinate system. In: Boundary Value Problems of Heat Conduction, International Textbook Company, Scranton, Pa, USA, pp: 125-193.

Pletcher, R.H., J.C. Tannehill and D. Anderson, 2012. Computational Fluid Mechanics and Heat Transfer. 3rd Edn., CRC Press, ISBN-10: 1591690374, pp: 774.

Vedat, S.A., 1966. Conduction Heat Transfer. 1st Edn., Addison-Wesley, USA, pp: 550.

Wang, H.M. and C.B. Liu, 2013. Analytical solution of two-dimensional transient heat conduction in fiber-reinforced cylindrical composites. Int. J. Thermal Sci., 69: 43-52. DOI: 10.1016/j.ijthermalsci.2013.02.001

Zalba, B., J.M. Marin, L.F. Cabeza and H. Mehlingc, 2003. Review on thermal energy storage with phase change: materials, heat transfer analysis and applications. Applied Thermal Eng., 23: 251-283. DOI: 10.1016/S1359-4311(02)00192-8

Zare, M. and S.A.G. Nassab, 2014. Combined radiative-conductive heat transfer in two-dimensional complex geometries with variable thermal conductivity. Iran. J. Sci. Technol., 38: 275-288.

Zhao, J., X. Ai, Y. Li and Y. Zhou, 2006. Thermal shock resistance of functionally gradient solid cylinders. Mater. Sci. Eng., 418: 99-110. DOI: 10.1016/j.msea.2005.11.019

Zill, D., W.S. Wright and M.R. Cullen, 2011. Advanced engineering mathematics. Jones and Bartlett Learning.