

Partial Eigenvalue Assignment for High-Order Linear Systems in a Time Delayed System

^{1,2}Ehab A. El-Sayed and ¹Eid E. El Behady

¹Department of Mathematics, College of Science and Humanitarian Studies, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia

²Department of Science and Engineering Mathematics, Faculty of Petroleum and Mining Engineering, Suez University, Egypt

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Corresponding Author:

Ehab A. El-Sayed

Department of Mathematics,

College of Science and

Humanitarian Studies, Prince

Sattam Bin Abdulaziz

University, Al-Kharj, Saudi

Arabia

Email: janine.tiu@gmail.com

Abstract: This paper introduces a solution to the partial eigenvalues assignment problem of high order linear systems with time delayed system in both single-input and multi-input cases using orthogonality relations between eigenvectors of the matrix polynomial. The solution requires the knowledge of only a few eigenvalues and with corresponding left eigenvectors of matrix polynomial. The numerical examples are done to illustrate the proposed method.

Keywords: Partial Eigenvalues Assignment Problem, Time Delayed, High-Order, Orthogonality Relations

Introduction

Consider high order dynamical system (Duan, 2005; Dedieu and Tisseur, 2003; Dedieu and Tisseur, 2001) of the form:

$$M_k \frac{d^k}{dt^k} v(t) + \dots + M_1 \frac{d}{dt} v(t) + M_0 v(t) = 0 \quad (1.1)$$

where, $\{M_k, M_{k-1}, \dots, M_1, M_0\}$ are real symmetric $n \times n$ matrices and M_k is positive definite and $v(t)$ is a time-dependent $n \times 1$ vector. Using $v(t) = x e^{\lambda t}$ in equation (1.1), where x is a real vector, then:

$$P(\lambda)x = (\lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0)x = 0 \quad (1.2)$$

Where matrix polynomial:

$$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0 \quad (1.3)$$

Is very often referred to as λ matrix, or matrix polynomial of degree k (Dedieu and Tisseur, 2003; Dedieu and Tisseur, 2001; Wang and Zhang, 2013; 2014; Ramadan and El-Sayed, 2010). λ (λ) is the characteristic roots of Equation (1.3), i.e., $\det(P(\lambda)) = 0$, are known as eigenvalues. The vectors $y \neq 0$ and $x \neq 0$ are corresponding left and right eigenvectors respectively which satisfy $y^H P(\lambda) = 0$ and $P(\lambda)x = 0$, where H is the conjugate transpose. The standard

eigenvalue problem $Ax = \lambda x$ is the special case of (1.2) for example, the references Wang and Zhang, 2013; 2014; Ramadan and El-Sayed, 2010).

The system modeled by (1.1) can be controlled with the application of a forcing function $Bu(t-\tau)$, $B \in R^{n \times m}$, ($m \leq n$) (B the control matrix) and $u(t-\tau)$ is a time-dependent $m \times 1$ real vector and τ is a constant time delay. The matrix B has full column rank, that is, $\text{rank}(B) = m$ in which case (1.1) is replaced by:

$$M_k \frac{d^k}{dt^k} v(t) + M_{k-1} \frac{d^{k-1}}{dt^{k-1}} v(t) + \dots + M_1 \frac{d}{dt} v(t) + M_0 v(t) = Bu(t-\tau) \quad (1.4)$$

With the choice of the controlling force:

$$u(t-\tau) = F_1^T \frac{d^{k-1}}{dt^{k-1}} v(t-\tau) + \dots + F_{k-1}^T \frac{d}{dt} v(t-\tau) + F_k^T v(t-\tau) \quad (1.5)$$

Such that, $F_1, F_2, \dots, F_k \in R^{m \times n}$ constant matrices, leads to the closed loop system:

$$M_k \frac{d^k}{dt^k} v(t) + M_{k-1} \frac{d^{k-1}}{dt^{k-1}} v(t) + \dots + M_1 \frac{d}{dt} v(t) + M_0 v(t) = B(F_1^T \frac{d^{k-1}}{dt^{k-1}} v(t-\tau) + \dots + F_{k-1}^T \frac{d}{dt} v(t-\tau) + F_k^T v(t-\tau)) \quad (1.6)$$

Differential Equation (1.6) leads, with the separation of variables $v(t) = xe^{\lambda t}$, to the problem of finding the eigenvalues of the modified matrix polynomial:

$$P_c(\lambda, \tau) = \lambda^k M_k + \dots + \lambda M_1 + M_0 - B(F_1^T \lambda^{k-1} + \dots + F_{k-1}^T \lambda + F_k^T) e^{-\lambda \tau} \quad (1.7)$$

The partial eigenvalue assignment problem for the time delayed system (1.6) is to find the control gain matrices F_1, F_2, \dots, F_k such that:

$$\det(P_c(\lambda_i, \tau)) = 0, \text{ for } i = 1, 2, \dots, kn \quad (1.8)$$

has the desired eigenvalues $\{\mu_1, \mu_2, \dots, \mu_m, \lambda_{m+1}, \dots, \lambda_{kn}\}$. That it contains m numbers of new assigned desired eigenvalues $\{\mu_1, \mu_2, \dots, \mu_m\}$ without affecting the remaining eigenvalues $\{\lambda_{m+1}, \dots, \lambda_{kn}\}$ of the Equation (1.1). Throughout this paper, notion (PEAP) is used instead of partial eigenvalue assignment problem.

PEAP arises from many practical situations such as electrical circuit simulation and acoustic system, vibration analysis of structural mechanical, fluid mechanics, finite element model updating in automobile industries and aerospace (Arévalo and Lötstedt, 1995; De Boor and Kreiss, 1986; Sand, 2002; Mackey *et al.*, 2006).

PEAP for high-order linear systems without time delay introduced in many paper (Wang and Zhang, 2014; Ramadan and El-Sayed, 2010). Prattand and Singh (2009) were solution PEAP for second order linear system with known time delay in single-input case. Also Singh *et al.* (2014) were solution the same problem but in Multi-input case and study its stability with known time delay. Wang and Zhang (2013) were introduced the direct method of PEAP for high order systems in single input case using receptance method with known time delay. In this paper, we proposed a method to solve PEAP for high order control systems in single input case and multi-input case with known time delay, using orthogonality relations between the eigenvectors of matrix polynomial. This solution requires only a partial knowledge of the eigenvalue and the corresponding eigenvector of the matrix polynomial which allows the partial assignment of desired eigenvalues with no spillover. The numerical examples illustrate that the proposed method.

Orthogonality Relations between the Eigenvectors of Matrix Polynomial

In this section, we introduce some Orthogonality relations which are play role important for solving our problem.

Theorem 1: (Sarkissian, 2001) Orthogonality of the Eigenvectors of a Matrix A

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $A \in C^{n \times n}$ and let \hat{X} and \hat{Y} be respectively the right and the left eigenvector matrices of A . Assume that $\{\lambda_1, \lambda_2, \dots, \lambda_m\} \cap \{\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_n\} = \emptyset$ and $m < n$. Partition $\hat{X} = (\hat{X}_1, \hat{X}_2)$ and $\hat{Y} = (\hat{Y}_1, \hat{Y}_2)$, where $\hat{X}_1 = (\hat{x}_1, \dots, \hat{x}_m)$, $\hat{X}_2 = (\hat{x}_{m+1}, \dots, \hat{x}_n)$, $\hat{Y}_1 = (\hat{y}_1, \dots, \hat{y}_m)$ and $\hat{Y}_2 = (\hat{y}_{m+1}, \dots, \hat{y}_n)$.

Then:

$$\hat{Y}_1^H \hat{X}_2 = 0 \quad (2.1)$$

And:

$$\hat{Y}_1^H A \hat{X}_2 = 0 \quad (2.2)$$

If, in addition, A is real symmetric, then:

$$\hat{Y}_1^H \hat{X}_2 = 0 \text{ and } \hat{Y}_1^H A \hat{X}_2 = 0 \quad (2.3)$$

The following theorems establish the orthogonality relations between the eigenvectors for the matrix polynomial using its connection with the standard eigenvalues problem.

Theorem 2 (Ramadan and El-Sayed, 2010) A scalar $\lambda \in C$ is an eigenvalue of the matrix polynomial

$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0$ with the corresponding right eigenvector x and the left eigenvector y if and only if λ is an eigenvalue of the $kn \times kn$ matrix:

$$A = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -M_k^{-1} M_0 & -M_k^{-1} M_1 & -M_k^{-1} M_2 & \dots & -M_k^{-1} M_{k-1} \end{pmatrix} \quad (2.4)$$

With the corresponding right eigenvector \hat{x} and left eigenvector \hat{y} such that:

$$\hat{x} = \begin{pmatrix} x \\ \lambda x \\ \lambda^2 x \\ \vdots \\ \lambda^{k-1} x \end{pmatrix} \text{ and } \hat{y} = \begin{pmatrix} (\lambda^{k-1} M_k^H + \lambda^{k-2} M_{k-1}^H + \dots + M_1^H) y \\ (\lambda^{k-2} M_k^H + \lambda^{k-3} M_{k-1}^H + \dots + M_2^H) y \\ (\lambda^{k-3} M_k^H + \lambda^{k-4} M_{k-1}^H + \dots + M_3^H) y \\ \vdots \\ (M_k^H) y \end{pmatrix} \quad (2.5)$$

Theorem 3 (Ramadan and El-Sayed, 2010)
(Orthogonality of the Eigenvectors of the Matrix Polynomial)

Let $\lambda_1, \lambda_2, \dots, \lambda_{kn}$ be the eigenvalues of the $kn \times kn$ matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0$ and let X and Y be respectively the right and left eigenvector matrices.

Assume that $\{\lambda_1, \dots, \lambda_m\} \cap \{\lambda_{m+1}, \dots, \lambda_{kn}\} = \emptyset$. Partition $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $A = \text{diag}(A_1, A_2)$ where $X_1 = (x_1, \dots, x_m)$, $X_2 = (x_{m+1}, \dots, x_{kn})$, $Y_1 = (y_1, \dots, y_m)$ and $Y_2 = (y_{m+1}, \dots, y_{kn})$ with $A_1 = \text{diag}(\lambda_1, \dots, \lambda_m)$ and $A_2 = \text{diag}(\lambda_{m+1}, \dots, \lambda_{kn})$.

Then:

$$\sum_{i=1}^{k-1} \left[\sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right] X_2 \Lambda_2^{k-i} - Y_1^H M_0 X_2 = 0 \quad (2.6)$$

And:

$$\sum_{i=1}^k \left[\sum_{j=1}^i [\Lambda_1^{j-1} Y_1^H M_{k-i+j}] \right] X_2 \Lambda_2^{k-i} = 0 \quad (2.7)$$

PEAP for High Order Control Systems in a Time Delayed System in Multi-Input Case

Write Equation (1.2) in the following form:

$$P(\lambda_i) x_i = \left(\lambda_i^k M_k + \lambda_i^{k-1} M_{k-1} \right) x_i = 0, i = 1, 2, \dots, kn \quad (3.1)$$

Or in the matrix form:

$$M_k X \Lambda^K + M_{k-1} X \Lambda^{K-1} + \dots + M_0 X = 0 \quad (3.2)$$

where, $X = (x_1, x_2, \dots, x_{kn}) \in C^{n \times kn}$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{kn}) \in C^{kn \times kn}$ where λ_i are all distinct. Where $\lambda_1, \dots, \lambda_{kn}$ are the eigenvalues of matrix polynomial:

$$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0 \quad (3.3)$$

Let us partition the $n \times kn$ right eigenvector matrix X , the $kn \times n$ left eigenvector matrix Y^H and $kn \times kn$ eigenvalues matrix Λ as follows:

$$X = (X_1, X_2), Y^H = \begin{pmatrix} Y_1^H \\ Y_2^H \end{pmatrix} \text{ and } \Lambda = \text{diag}(\Lambda_1, \Lambda_2) \quad (3.4)$$

where, $X_1 = (x_1, \dots, x_m)$, $X_2 = (x_{m+1}, \dots, x_{kn})$, $Y_1 = (y_1, \dots, y_m)$ and $Y_2 = (y_{m+1}, \dots, y_{kn})$ with $A_1 = \text{diag}(\lambda_1, \dots, \lambda_m)$ and $A_2 = \text{diag}(\lambda_{m+1}, \dots, \lambda_{kn})$.

Given m complex numbers $\mu_1, \mu_2, \dots, \mu_m$ closed under complex conjugation, $m < n$ and a matrix $B \in R^{n \times m}$ with

known time delay τ , we are required to find $F_1, F_2, \dots, F_k \in R^{n \times m}$ such that the modified matrix polynomial:

$$P_c(\lambda, \tau) = \lambda^k M_k + \dots + \lambda M_1 + M_0 - B \left(F_1^T \lambda^{k-1} + \dots + F_{k-1}^T \lambda + F_k^T \right) e^{-\lambda \tau} \quad (3.5)$$

has spectrum $\{\mu_1, \mu_2, \dots, \mu_m, \lambda_{m+1}, \dots, \lambda_{kn}\}$. PEAP for high-order linear systems in Multi-input case with time delay in which we use the matrices $F_1, F_2, \dots, F_k \in R^{n \times m}$ to replace the eigenvalues $\{\lambda_j\}_{j=1}^m$ of the matrix polynomial

$P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \dots + \lambda M_1 + M_0$ by $\{\mu_j\}_{j=1}^m$, while leaving the other eigenvalues unchanged. Next, we introduce the following theorem for solving PEAP of high order linear systems with time delay.

Theorem 4

Let the feedback matrices $\{F_i\}_{i=1}^{k-1}$ and F_k be defined by:

$$F_i = \sum_{j=1}^i \left[M_{k-i+j}^T \bar{Y}_1^H \Lambda_1^j \right] \varphi^T, \quad (3.6)$$

$$F_k = -M_0^T \bar{Y}_1^H \varphi^T, i = 1, 2, \dots, k-1, \varphi \in C^{m \times m}$$

Then for any choice of φ we have:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - \left(B F_1^T X_2 \Lambda_2^{k-1} + \dots + B F_k^T X_2 \right) e^{-\Lambda_2 \tau} = 0 \quad (3.7)$$

In words, this theorem assures us that any choice of φ with the feedback matrices $\{F_i\}_{i=1}^k$ as in (3.6) guarantees that the last $kn-m$ eigenpairs (A_2, X_2) of the matrix polynomial $P(\lambda)$ are also eigenpairs of the modified matrix polynomial $P_c(\lambda, \tau)$.

Proof

Our goal is to prove that:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - \left(B F_1^T X_2 \Lambda_2^{k-1} + \dots + B F_k^T X_2 \right) e^{-\Lambda_2 \tau} = 0 \quad (3.8)$$

Expanding the left hand side of (3.8) by substituting (3.6) in (3.8), we obtain:

$$\begin{aligned} & M_k X_2 \Lambda_2^k + \dots + M_0 X_2 \\ & - B \left(\varphi \sum_{i=1}^{k-1} \sum_{j=1}^i \left[\Lambda_1^j Y_1^H M_{k-i+j} \right] \right) e^{-\Lambda_2 \tau} \\ & \left(X_2 \Lambda_2^{k-i} - \varphi Y_1^H M_0 X_2 \right) \\ & = -B \varphi \left(\sum_{i=1}^{k-1} \sum_{j=1}^i \left[\Lambda_1^j Y_1^H M_{k-i+j} \right] \right) e^{-\Lambda_2 \tau} \\ & \left(X_2 \Lambda_2^{k-i} - Y_1^H M_0 X_2 \right) \end{aligned} \quad (3.9)$$

Since (X_2, Λ_2) is an eigenpair of the open loop pencil (3.2) we have:

$$M_k X_2 \Lambda_2^k + M_{k-1} X_2 \Lambda_2^{k-1} + \dots + M_0 X_2 = 0 \quad (3.10)$$

And furthermore:

$$\sum_{i=1}^{k-1} \left[\sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right] X_2 \Lambda_2^{k-i} - Y_1^H M_0 X_2 = 0 \quad (3.11)$$

From theorem 3. Thus:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - (BF_1^T X_2 \Lambda_2^{k-1} + \dots + BF_k^T X_2) e^{-\Lambda_2 \tau} = 0 \quad (3.12)$$

The theorem is then proved.

In order to use theorem 4 to solve PEAP, we need to choose φ which will change eigenvalues $\{\lambda_j\}_{j=1}^m$ of the matrix polynomial $P(\lambda)$ to $\{\mu_j\}_{j=1}^m$ in $P_c(\lambda, \tau)$, if that is possible. If there is such the matrix φ^T , then there exist an eigenvector matrix $Z \in \mathbb{C}^{m \times m}$ such that $Z = (z_1, z_2, \dots, z_m)$, $z_j \neq 0, j = 1, 2, \dots, m$ and matrix $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_m)$ and $e^{-D\tau} = \text{diag}(e^{-\mu_1 \tau}, e^{-\mu_2 \tau}, \dots, e^{-\mu_m \tau})$ which are such that:

$$M_k Z D^k + \dots + M_0 Z - (BF_1^T Z D^{k-1} + \dots + BF_k^T Z) e^{-D\tau} = 0 \quad (3.13)$$

Substituting $\{F_i\}_{i=1}^k$ form (3.6) in (3.13) and after rearranging, we obtain:

$$M_k Z D^k + M_{k-1} Z D^{k-1} + \dots + M_0 Z = B\varphi \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right) e^{-D\tau} = B\varphi W^H \quad (3.14)$$

Then:

$$M_k Z D^k + M_{k-1} Z D^{k-1} + \dots + M_0 Z = B\Gamma \quad (3.15)$$

where, $W^H = \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right) Z D^{k-i} - Y_1^H M_0 Z$ and

$W\varphi^H = \Gamma^H$ is matrix that will depend on the scaling chosen for the eigenvectors in Z . To obtain Z , we choosing the matrix Γ in the following form $\Gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$. Then the equation (3.15) becomes:

$$M_k Z D^k + M_{k-1} Z D^{k-1} + \dots + M_0 Z = B(\gamma_1, \gamma_2, \dots, \gamma_m) \quad (3.16)$$

We can solve for each of the eigenvectors z_j using the equations in such a way that $\mu_j = \bar{\mu}_i$ implies $\gamma_j = \bar{\gamma}_i$ for $i = 1, 2, \dots, m$ and solve for z_1, \dots, z_m :

$$(M_k \mu_j^k + M_{k-1} \mu_j^{k-1} + \dots + M_0) z_j = B \gamma_j, j = 1, 2, \dots, m \quad (3.17)$$

So, we computed the eigenvectors Z and hence we computed the matrix W from:

$$W^H = \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right) Z D^{k-i} - Y_1^H M_0 Z \quad (3.18)$$

We solve the $m \times m$ square linear system for φ :

$$W\varphi^H = (\gamma_1, \gamma_2, \dots, \gamma_m)^H \quad (3.19)$$

Finally, determine the control gain matrices $F_i, i = 1, 2, \dots, k$ are obtained from (3.6).

PEAP for High Order Control Systems in a Time Delayed System in Single-Input Case

For single-input control application, the system (1.1) is modified by applying a controlling force $bu(t-\tau)$ as:

$$M_k \frac{d^k}{dt^k} v(t) + M_{k-1} \frac{d^{k-1}}{dt^{k-1}} v(t) + \dots + M_1 \frac{d}{dt} v(t) + M_0 v(t) = bu(t-\tau) \quad (4.1)$$

where, $b \in \mathbb{R}^{n \times 1}$, (b is the control vector) and control force $u(t-\tau)$, is defined as:

$$u(t-\tau) = f_1^T \frac{d^{k-1}}{dt^{k-1}} v(t-\tau) + \dots + f_{k-1}^T \frac{d}{dt} v(t-\tau) + f_k^T v(t-\tau) \quad (4.2)$$

Such that, f_1, f_2, \dots, f_k vectors, leads to the closed loop system:

$$M_k \frac{d^k}{dt^k} v(t) + M_{k-1} \frac{d^{k-1}}{dt^{k-1}} v(t) + \dots + M_1 \frac{d}{dt} v(t) + M_0 v(t) = b(f_1^T \frac{d^{k-1}}{dt^{k-1}} v(t-\tau) + \dots + f_{k-1}^T \frac{d}{dt} v(t-\tau) + f_k^T v(t-\tau)) \quad (4.3)$$

where, τ is the known time delay with the separation of variables $v(t) = xe^{\lambda t}$, to the problem of finding the eigenvalues of the modified matrix polynomial:

$$P_c(\lambda, \tau) = \lambda^k M_k + \dots + \lambda M_1 + M_0 - b(f_1^T \lambda^{k-1} + \dots + f_{k-1}^T \lambda + f_k^T) e^{-\lambda \tau} \quad (4.4)$$

PEAP with the time delayed system (4.4) is to find the control vectors f_1, f_2, \dots, f_k such that $\det(P_c(\lambda, \tau)) = 0$,

for $i = 1, 2, \dots, kn$, has the desired eigenvalues $\{\mu_1, \mu_2, \dots, \mu_m, \lambda_{m+1}, \dots, \lambda_{kn}\}$. That it contains m numbers of newly assign need desired eigenvalues $\mu_1, \mu_2, \dots, \mu_m$ without affecting the remaining eigenvalues $\{\lambda_{m+1}, \dots, \lambda_{kn}\}$ of the open loop system (1.1).

Theorem 5

Let the feedback vectors $\{f_i\}_{i=1}^{k-1}$ and f_k be defined by:

$$f_i = \sum_{j=1}^i [M_{k-i+j}^T \bar{Y}_1 \Lambda_1^j] \varphi^T, f_k = -M_0^T \bar{Y}_1 \varphi^T, \quad (4.5)$$

$i = 1, 2, \dots, k-1. \varphi \in C^{1 \times m}$

Then for any choice of φ the following relation holds:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - (b f_1^T X_2 \Lambda_2^{k-1} + \dots + b f_k^T X_2) e^{-\Lambda_2 \tau} = 0 \quad (4.6)$$

Proof

Our goal is to prove that:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - (b f_1^T X_2 \Lambda_2^{k-1} + \dots + b f_k^T X_2) e^{-\Lambda_2 \tau} = 0 \quad (4.7)$$

Expanding the left hand side of (4.7) by substituting (4.5) in (4.7), we obtain:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - b \left(\varphi \sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right) e^{-\Lambda_2 \tau} - (X_2 \Lambda_2^{k-i} - \varphi Y_1^H M_0 X_2) e^{-\Lambda_2 \tau} = -b \varphi \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] X_2 \Lambda_2^{k-i} - Y_1^H M_0 X_2 \right) e^{-\Lambda_2 \tau} \quad (4.8)$$

Since (X_2, Λ_2) is an eigenpair of the open loop pencil we have:

$$M_k X_2 \Lambda_2^k + M_{k-1} X_2 \Lambda_2^{k-1} + \dots + M_0 X_2 = 0 \quad (4.9)$$

And furthermore:

$$\sum_{j=1}^{k-1} \left[\sum_{i=1}^j [\Lambda_1^i Y_1^H M_{k-i+j}] \right] X_2 \Lambda_2^{k-i} - Y_1^H M_0 X_2 = 0 \quad (4.10)$$

From theorem 3. Thus:

$$M_k X_2 \Lambda_2^k + \dots + M_0 X_2 - (b f_1^T X_2 \Lambda_2^{k-1} + \dots + b f_k^T X_2) e^{-\Lambda_2 \tau} = 0 \quad (4.11)$$

The theorem is then proved.

From theorem 5, we need to choose φ which will change eigenvalues $\{\lambda_j\}_{j=1}^m$ of the matrix polynomial $P(\lambda)$ to $\{\mu_j\}_{j=1}^m$ in $P_c(\lambda, \tau)$, if that is possible. If there is such the matrix φ^T , then there exist an eigenvector matrix $Z \in C^{m \times m}$ such that $Z = (z_1, z_2, \dots, z_m), z_j \neq 0, j = 1, 2, \dots, m$ and matrix $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_m)$ and $e^{-D\tau} = \text{diag}(e^{-\mu_1 \tau}, e^{-\mu_2 \tau}, \dots, e^{-\mu_m \tau})$ which are such that:

$$M_k Z D^k + \dots + M_0 Z - (b f_1^T Z D^{k-1} + \dots + b f_k^T Z) e^{-D\tau} = 0 \quad (4.12)$$

Substituting $\{f_i\}_{i=1}^k$ form (4.5) in (4.12) and after rearranging, we obtain:

$$M_k Z D^k + \dots + M_0 Z = b \varphi \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] \right) e^{-D\tau} = b \varphi W^H \quad (4.13)$$

Then:

$$M_k Z D^k + M_{k-1} Z D^{k-1} + \dots + M_0 Z = b C \quad (4.14)$$

where, $W^H = \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] Z D^{k-i} - Y_1^H M_0 Z \right) e^{-D\tau}$ and $W \varphi^H = C^H$ is matrix that will depend on the scaling chosen for the eigenvectors in Z . To obtain Z , we choosing the vector C in the following $C = (1, 1, \dots, 1)$. Then the equation (4.14) becomes:

$$M_k Z D^k + M_{k-1} Z D^{k-1} + \dots + M_0 Z = b(1, 1, \dots, 1) \quad (4.15)$$

We can solve for each of the eigenvectors z_j :

$$(M_k \mu_j^k + M_{k-1} \mu_j^{k-1} + \dots + M_0) z_j = b, j = 1, 2, \dots, m \quad (4.16)$$

So, we computed the eigenvectors Z and hence we computed the matrix W from:

$$W^H = \left(\sum_{i=1}^{k-1} \sum_{j=1}^i [\Lambda_1^j Y_1^H M_{k-i+j}] Z D^{k-i} - Y_1^H M_0 Z \right) e^{-D\tau} \quad (4.17)$$

We solve the $m \times 1$ linear system for φ :

$$W \varphi^H = (1, 1, \dots, 1)^T \quad (4.18)$$

Finally, determine the control vectors $f_i, i = 1, 2, \dots, k$ are obtained from (4.5).

Numerical Examples

Example (1) (Multi-Input Case)

Let the randomly matrices M_3, M_2, M_1 and M_0 (size 4) as follows:

$$M_3 = \begin{bmatrix} 0.9501 & 0.8913 & 0.8214 & 0.9218 \\ 0.2311 & 0.7621 & 0.4447 & 0.7382 \\ 0.6068 & 0.4565 & 0.6154 & 0.1763 \\ 0.4860 & 0.0185 & 0.7919 & 0.4057 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0.3046 & 0.3028 & 0.3784 & 0.4966 \\ 0.1897 & 0.5417 & 0.8600 & 0.8998 \\ 0.1934 & 0.1509 & 0.8537 & 0.8216 \\ 0.6822 & 0.6979 & 0.5936 & 0.6449 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0.4451 & 0.8462 & 0.8381 & 0.8318 \\ 0.9318 & 0.5252 & 0.0196 & 0.5028 \\ 0.4660 & 0.2026 & 0.6813 & 0.7095 \\ 0.4186 & 0.6721 & 0.3795 & 0.4289 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 0.9355 & 0.0579 & 0.1389 & 0.2722 \\ 0.9169 & 0.3529 & 0.2028 & 0.1988 \\ 0.4103 & 0.8132 & 0.1987 & 0.0153 \\ 0.8936 & 0.0099 & 0.6038 & 0.7468 \end{bmatrix}$$

And a randomly matrix B (the control matrix), is:

$$B = \begin{bmatrix} 8.1797e-001 & 8.1797e-001 \\ 3.4197e-001 & 2.8973e-001 \\ 3.4119e-001 & 5.3408e-001 \\ 7.2711e-001 & 3.0929e-001 \end{bmatrix}$$

And $\tau = 0.1$ eigenvalues of $P(\lambda) = M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0$ are: $\lambda_{1,2} = 1.1687 \pm 0.8481i$, $\lambda_{3,4} = 0.0375 \pm 0.9101i$, $\lambda_{5,6} = -0.2044 \pm 0.5375i$, $\lambda_7 = 0.1275$, $\lambda_8 = 0.6132$, $\lambda_9 = 0.9085$, $\lambda_{10} = -0.9321$, $\lambda_{11} = -1.6155$, $\lambda_{12} = -2.1822$.

Now, we reassign the first two $m = 2$ eigenvalues $\lambda_{1,2} = 1.1687 \pm 0.8481i$ to the conjugate pair $\mu_{1,2} = -1 \pm i$.

From theorem 3, we obtain $A_1 = \text{diag}(1.1687 + 0.8481i, 1.1687 - 0.8481i)$:

$$Y_1^H = \begin{bmatrix} -0.1014 + 0.6844i & -0.1014 - 0.6844i \\ 0.0808 - 0.3962i & 0.0808 + 0.3962i \\ -0.6505 - 0.3211i & -0.6505 + 0.3211i \\ 0.7606 - 0.0701i & 0.7606 + 0.0701i \end{bmatrix}$$

And $D = \text{diag}(-1+i, -1-i)$. By following the procedure in the section 3 and assume that:

$$\Gamma = (\gamma_1, \gamma_2) = \begin{bmatrix} 0.83850 - 0.37041i & 0.56807 - 0.70274i \\ 0.83850 + 0.37041i & 0.56807 + 0.70274i \end{bmatrix}$$

We compute:

$$W = \begin{bmatrix} 0.0350 + 0.1648i & -0.0791 - 0.3293i \\ -0.0791 + 0.3293i & 0.0350 - 0.1648i \end{bmatrix}$$

And:

$$\varphi = \begin{bmatrix} 1.0117 - 1.7874i & 3.2719 - 1.4419i \\ 1.0117 + 1.7874i & 3.2719 + 1.4419i \end{bmatrix}$$

From which we compute the feedback matrices, F_1, F_2 and F_3 in view of (3.6):

$$F_1 = \begin{bmatrix} -1.8746 & -3.4758 \\ -0.7359 & -3.0500 \\ -0.8857 & -0.3800 \\ -1.5998 & -1.4270 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -1.8799 & -2.9836 \\ -0.7359 & 1.3408 \\ 0.6590 & 0.9684 \\ -0.5921 & -1.2031 \end{bmatrix}$$

And

$$F_3 = \begin{bmatrix} -0.4984 & -2.3275 \\ -0.2848 & 2.2204 \\ -0.9992 & -2.4381 \\ -0.9263 & -3.4316 \end{bmatrix}$$

Once the control gain matrices are obtained, by computing:

$$\det(P_c(\lambda_i, \tau)) = \det \left(\begin{bmatrix} \lambda_i^3 M_3 + \lambda_i^2 M_2 + \lambda_i M_1 + M_0 - B \\ (F_1^T \lambda_i^2 + F_2^T \lambda_i + F_3^T) e^{-\lambda_i \tau} \end{bmatrix} \right)$$

For $i = 1, 2, \dots, 12$. Where $\lambda_i \in \{\mu_1, \mu_2, \lambda_3, \lambda_4, \dots, \lambda_{12}\}$, it is verified that $|\det(P_c(\lambda_i, \tau))| < 10^{-12}$ validating the desired partial eigenvalue assignment.

Example (2) (Single-Input Case)

Let the randomly matrices M_3, M_2, M_1 and M_0 (size 4) as in example 1 and a randomly vector b (the control vector):

$$b = \begin{bmatrix} 0.8180 \\ 0.6602 \\ 0.3420 \\ 0.2897 \end{bmatrix} \text{ and } \tau = 0.1$$

Eigenvalues of $P(\lambda) = M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0$ are: $\lambda_{1,2} = 1.1687 \pm 0.8481i$, $\lambda_{3,4} = 0.0375 \pm 0.9101i$, $\lambda_{5,6} = -0.2044 \pm 0.5375i$, $\lambda_7 = 0.1275$, $\lambda_8 = 0.6132$, $\lambda_9 = 0.9085$, $\lambda_{10} = -0.9321$, $\lambda_{11} = -1.6155$, $\lambda_{12} = -2.1822$.

Now, we reassign the first two $m = 2$ eigenvalues $\lambda_{1,2} = 1.1687 \pm 0.8481i$ to the conjugate pair $\mu_{1,2} = -1 \pm i$.

From theorem 3, we obtain $A_1 = \text{diag}(1.1687 + 0.8481i, 1.1687 - 0.8481i)$:

$$Y_1^H = \begin{bmatrix} -0.1014 + 0.6844i & -0.1014 - 0.6844i \\ 0.0808 - 0.3962i & 0.0808 + 0.3962i \\ -0.6505 - 0.3211i & -0.6505 + 0.3211i \\ 0.7606 - 0.0701i & 0.7606 + 0.0701i \end{bmatrix}$$

And $D = \text{diag}(-1+i, -1-i)$. By the procedure in the section 4 we compute:

$$W = \begin{bmatrix} 0.0198 - 0.0937i & 0.0887 + 0.0889i \\ 0.0887 - 0.0889i & 0.0198 + 0.0937i \end{bmatrix}$$

$$\text{and } \varphi = \begin{bmatrix} 10.4513 - 0.7334i \\ 10.4513 + 0.7334i \end{bmatrix}$$

We compute the feedback vectors, f_1, f_2 and f_3 in view of (4.5):

$$f_1 = \begin{bmatrix} -7.9192 \\ -10.5668 \\ 1.8435 \\ 0.0530 \end{bmatrix}, f_2 = \begin{bmatrix} -5.7201 \\ 1.5189 \\ 1.6624 \\ -2.9669 \end{bmatrix} \text{ and } f_3 = \begin{bmatrix} -8.3155 \\ 10.9586 \\ -6.8112 \\ -11.4977 \end{bmatrix}$$

Once the control gain vectors are obtained, by computing:

$$\det(P_c(\lambda_i, \tau)) = \det \left(\begin{array}{c} \lambda_i^3 M_3 + \lambda_i^2 M_2 + \lambda_i M_1 + M_0 - b \\ (f_1^T \lambda_i^2 + f_2^T \lambda_i + f_3^T) e^{-\lambda_i \tau} \end{array} \right)$$

For $i = 1, 2, \dots, 12$. Where $\lambda_i \in \{\mu_1, \mu_2, \lambda_3, \lambda_4, \dots, \lambda_{12}\}$, it is verified that $|\det(P_c(\lambda_i, \tau))| < 10^{-12}$ validating the desired partial eigenvalue assignment.

Conclusion

We derived a solution to PEAP for high-order linear systems (1.4) in both multi-input and single-input case by using orthogonality relations between eigenvectors for matrix polynomial $P(\lambda)$. The approach needs the knowledge of only a few eigenvalues and with corresponding left eigenvectors of matrix polynomial $P(\lambda)$ without turning high order systems into first order form. When $\tau = 0$ is a special case of our results see the reference (Ramadan and El-Sayed, 2010).

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Author's Contributions

Ehab A. El-Sayed: Participated in all experiments, coordinated the data-analysis and contributed to the writing of the manuscript.

Eid E. El Behady: Designed the research plan and organized the study and contributed to the writing of the manuscript.

Ethics

Department of Mathematics, College of Science and Humanitarian Studies, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia.

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